Multivariable Processes

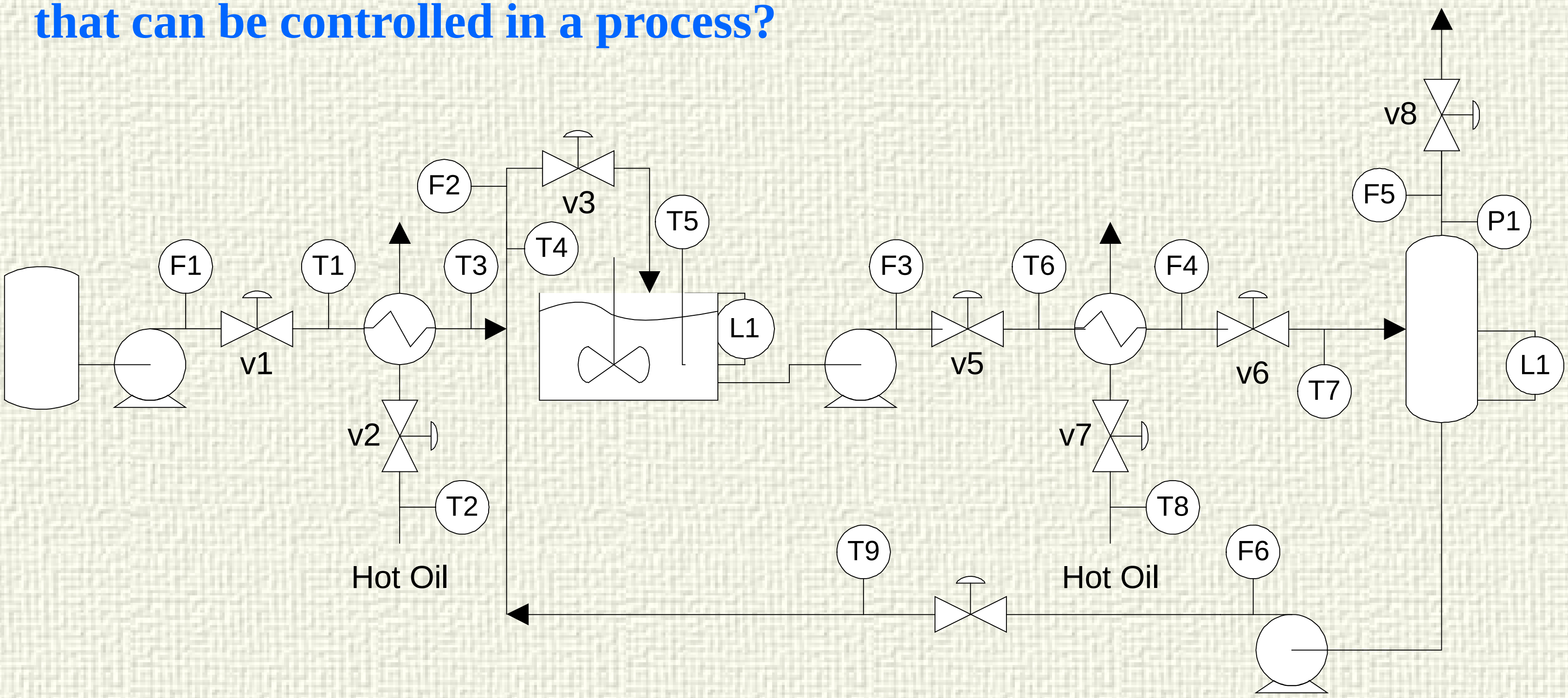
Prof. Cesar de Prada
ISA-UVA
prada@autom.uva.es

Outline

- Interaction
- Control of multivariable processes using SISO controllers
- RGA
- Control loop pairing
- Decoupling Control
- Multivariable Control

Degrees of freedom

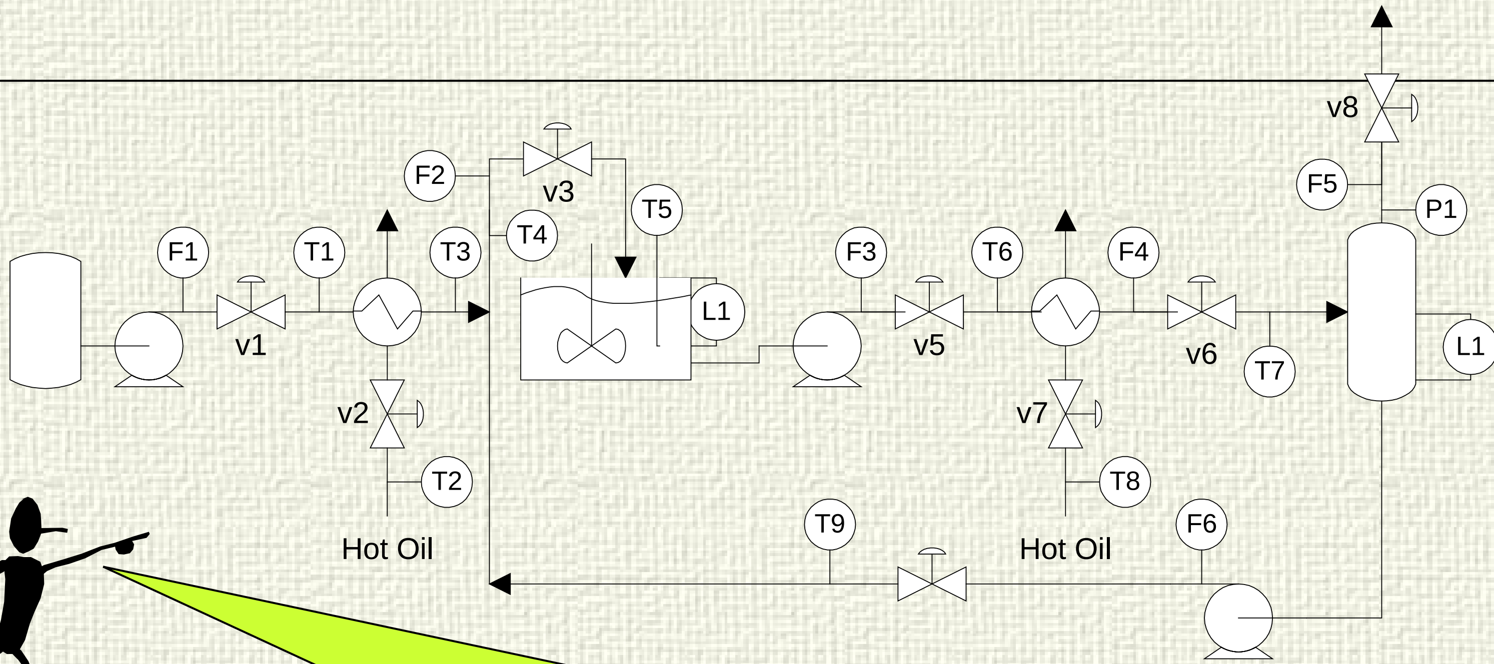
How to determine the maximum number of variables that can be controlled in a process?

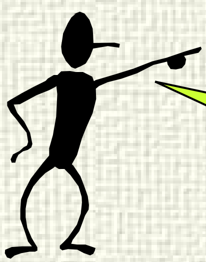


Degrees of freedom

A basic requirement :

Number of valves (actuators) ≥ number of controlled variables

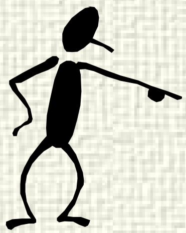




This is a necessary but not sufficient condition in order to satisfy the control aims!

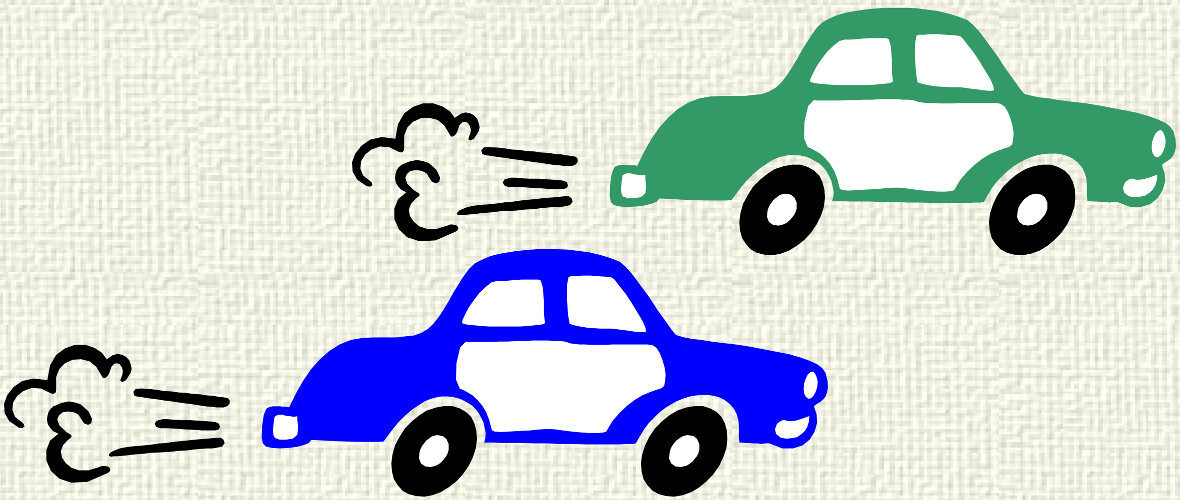
Interaction

An example!!!



<u>These cars</u>

- **Are they controllable independently?**
- **Does it exist interaction?**



Interaction

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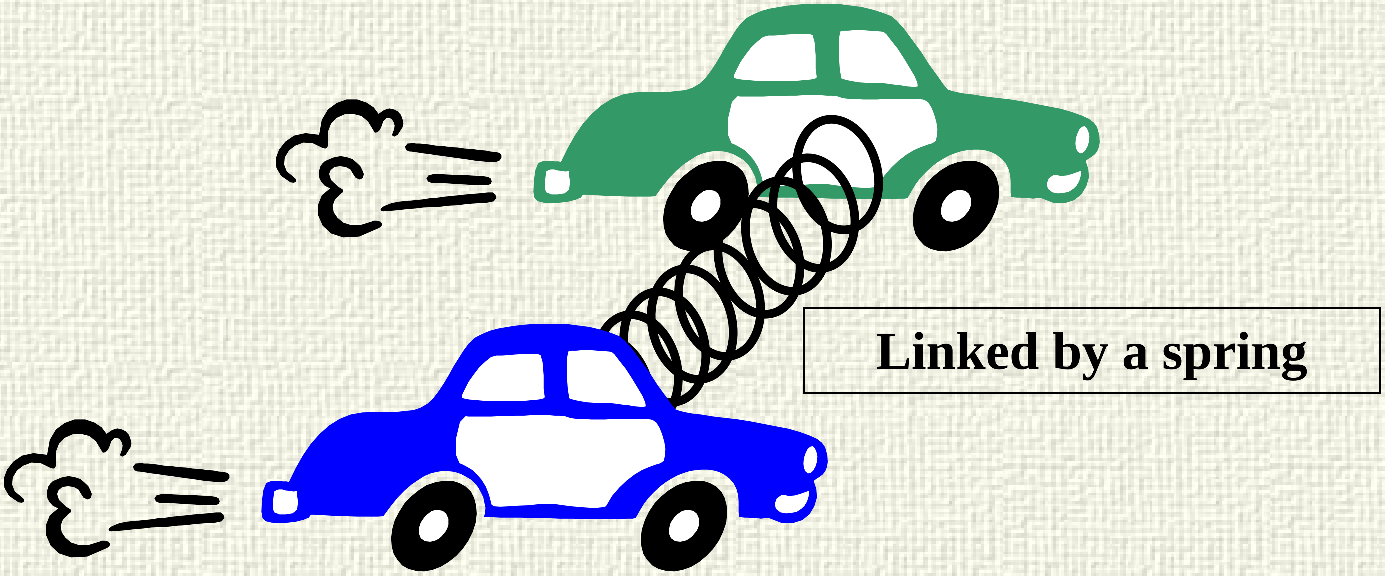

Interaction

<u>**These cars**</u>

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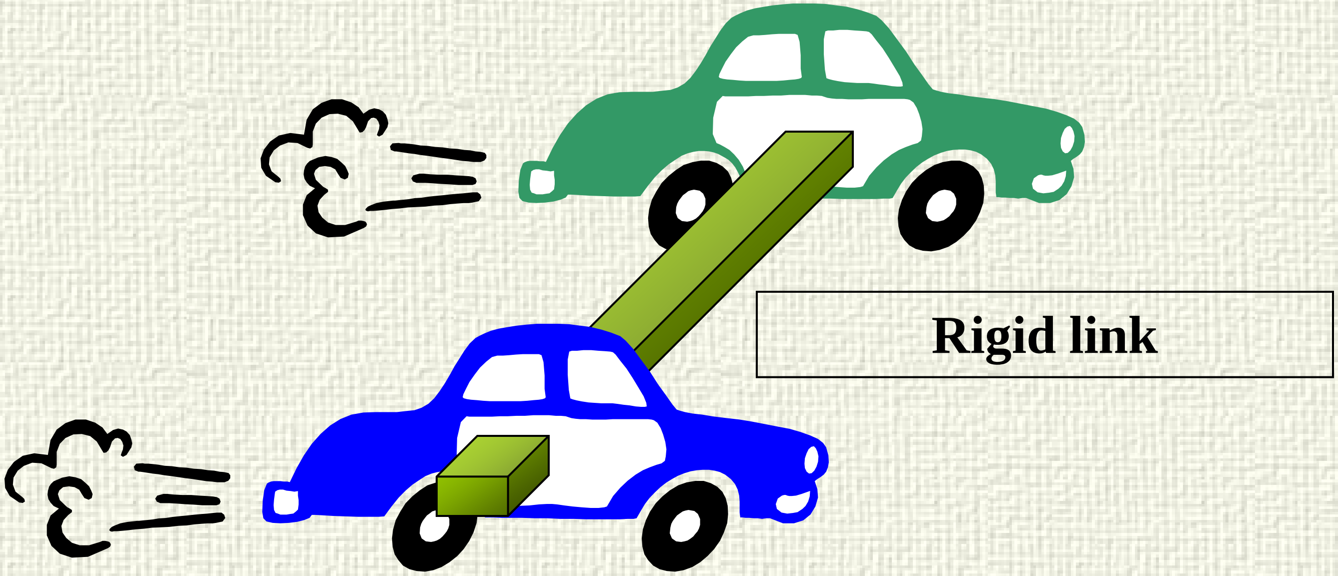
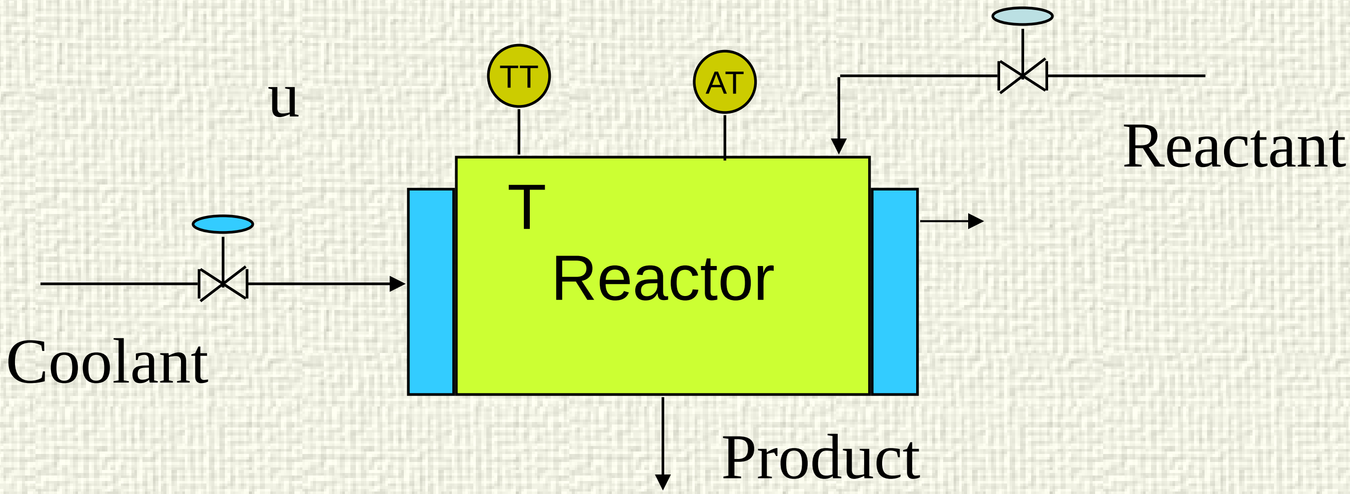


Reactor

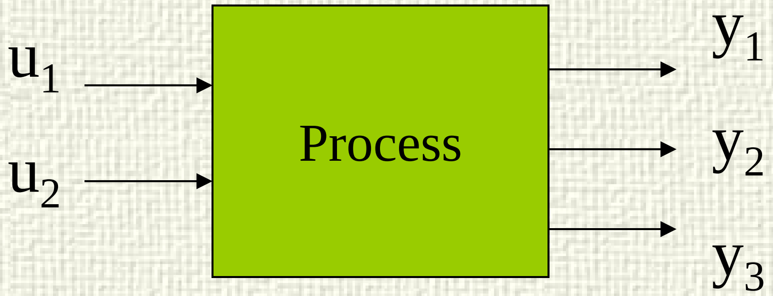


Input output interaction in both variables

Open loop interaction / Not necessarily equal to closed loop interaction

MIMO (Multi Input Multi Output) Systems

Independent Manipulated variables



Dependent Controlled variables

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \\ Y_3(s) \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \\ G_{31}(s) & G_{32}(s) \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix}$$

Interaction

Directions

Controlability/Operability

A process is said to be controllable/operable if the controlled variables can be kept in its set points in steady state, in spite of the disturbances acting on the plant

Model of a 2x2 process

$$\begin{bmatrix} CV_1 \\ CV_2 \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} MV_1 \\ MV_2 \end{bmatrix} + \begin{bmatrix} K_{d1} \\ K_{d2} \end{bmatrix} D$$

Mathematically, for a process to be controllable, the gain matrix of the process should be able to be inverted, that is, its determinant should be $K \neq 0$.

Controlability/Operability

A process is said to be controllable/operable if the controlled variables can be kept in its set points in steady state, in spite of the disturbances acting on the plant

Model of a 2x2 process

$$\begin{bmatrix} MV_1 \\ MV_2 \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}^{-1} \left[\begin{bmatrix} CV_1 \\ CV_2 \end{bmatrix} - \begin{bmatrix} K_{d1} \\ K_{d2} \end{bmatrix} D \right]$$

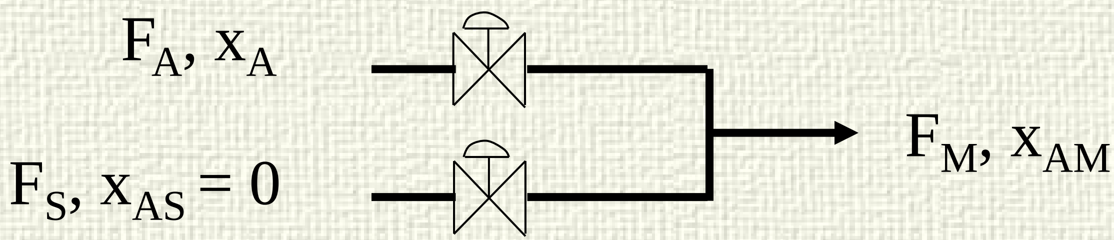
If the determinant K ≠ 0, then we can find values of the MV's that maintain the CV's on spite of the value of the DV's. (Assuming they remain within the appropriate range).

Controlability

In this blending process

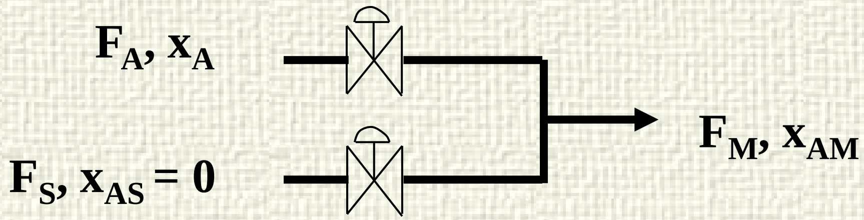
- **Can F_M and x_{AM} be controlled independently?**
- **Is there interaction among the process variables ?**

x_A = cte.



$$F_M = F_A + F_S \quad \Rightarrow \quad \Delta F_M = \Delta F_A + \Delta F_S$$

$$x_{AM} = \frac{F_A x_A}{F_A + F_S} \quad \Rightarrow \quad \Delta x_{AM} = \left[\frac{(1 - x_A)F_A}{(F_s + F_A)^2}\right]_{ss} \Delta F_A + \left[\frac{-F_A x_A}{(F_s + F_A)^2}\right]_{ss} \Delta F_S$$



$$\begin{bmatrix}\Delta F_M \\ \Delta x_{AM}\end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \left[\dfrac{(1-x_A)F_A}{(F_s+F_A)^2}\right]_{ss} & \left[\dfrac{-F_A x_A}{(F_s+F_A)^2}\right]_{ss}\end{bmatrix}\begin{bmatrix}\Delta F_A \\ \Delta F_S\end{bmatrix}$$

$$Det(K) = \frac{-F_A x_A}{(F_A+F_S)^2} - \frac{F_A(1-x_A)}{(F_A+F_S)^2} = \frac{-F_A}{(F_A+F_S)^2} \neq 0$$

Yes, the process is controllable!

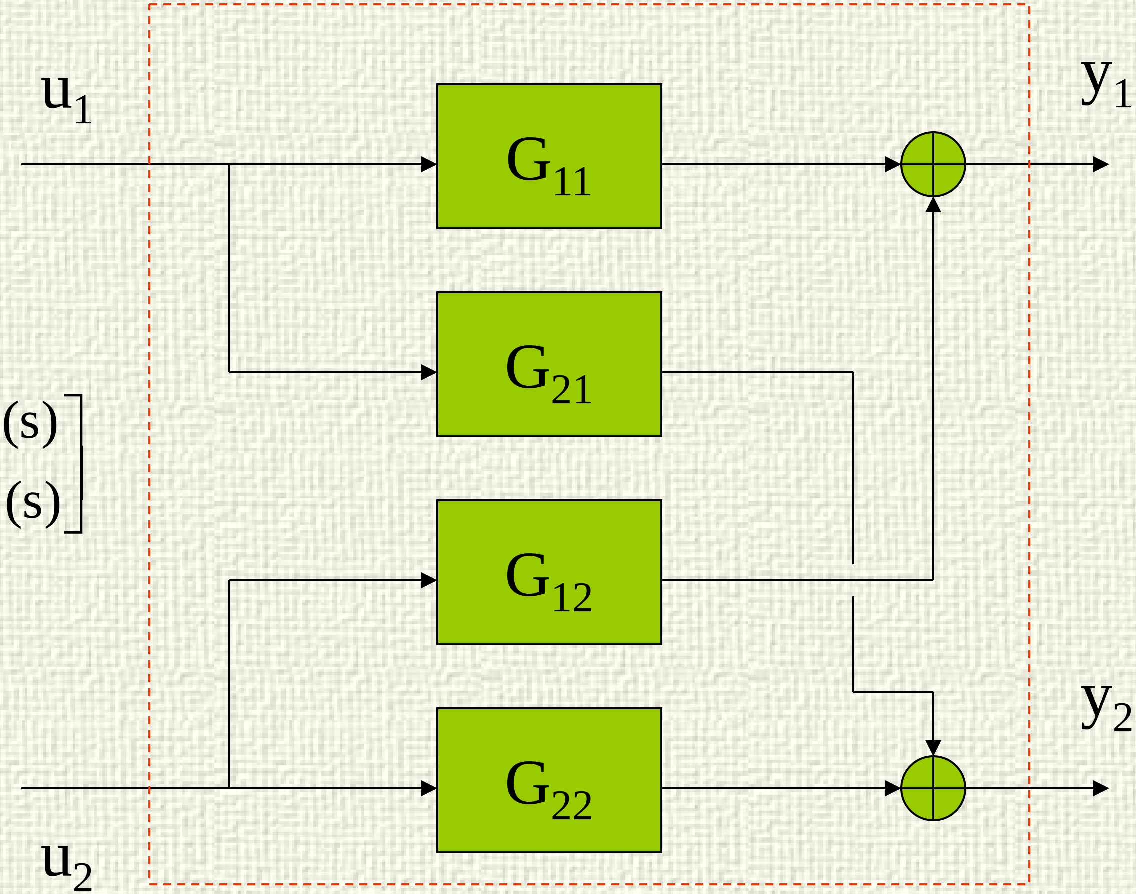
Would it be controllable if x_{AS} **were different from zero?**

Interaction

Open loop

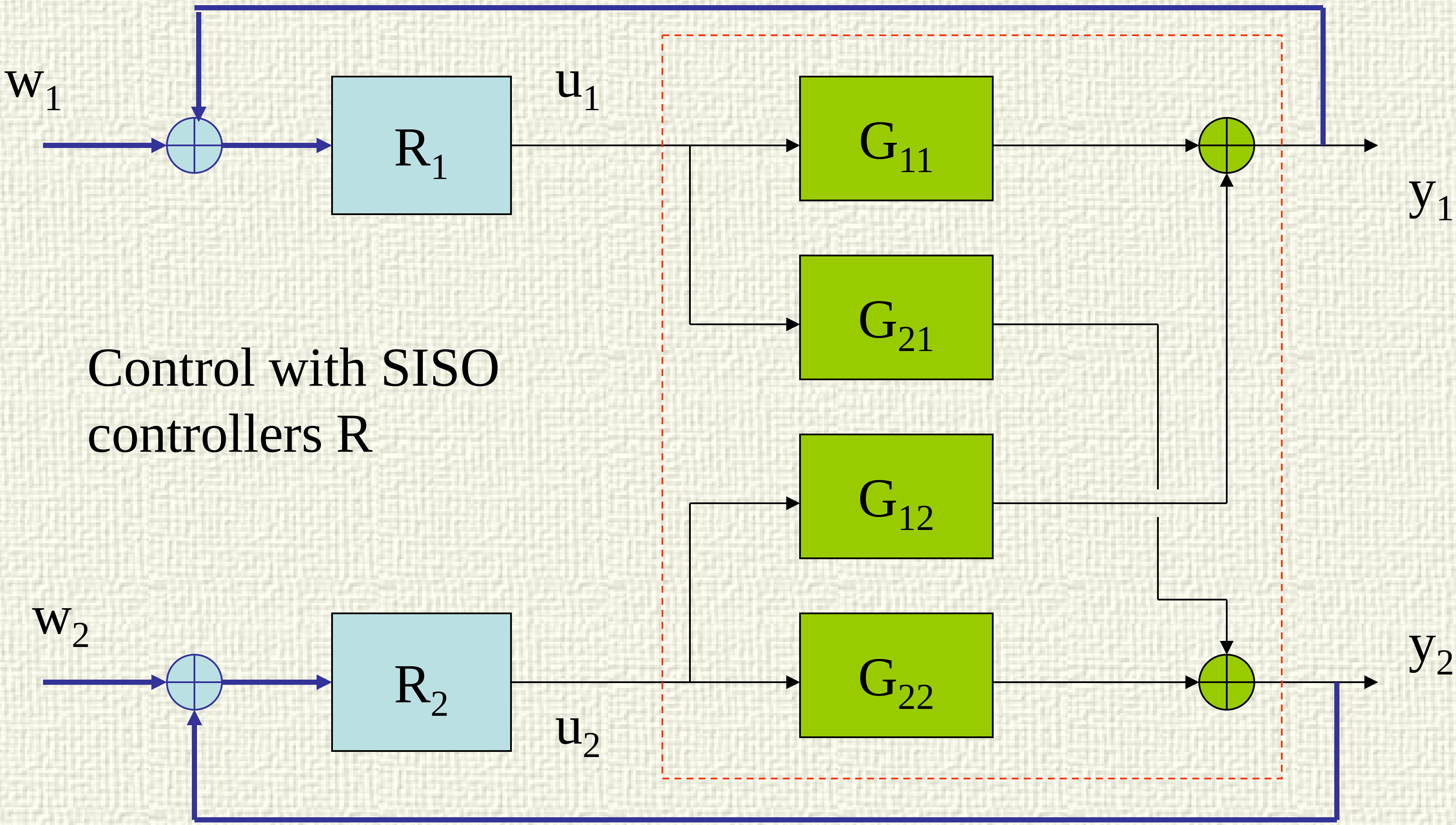
$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} =$$

$$= \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix}$$



How does it behave in closed loop?

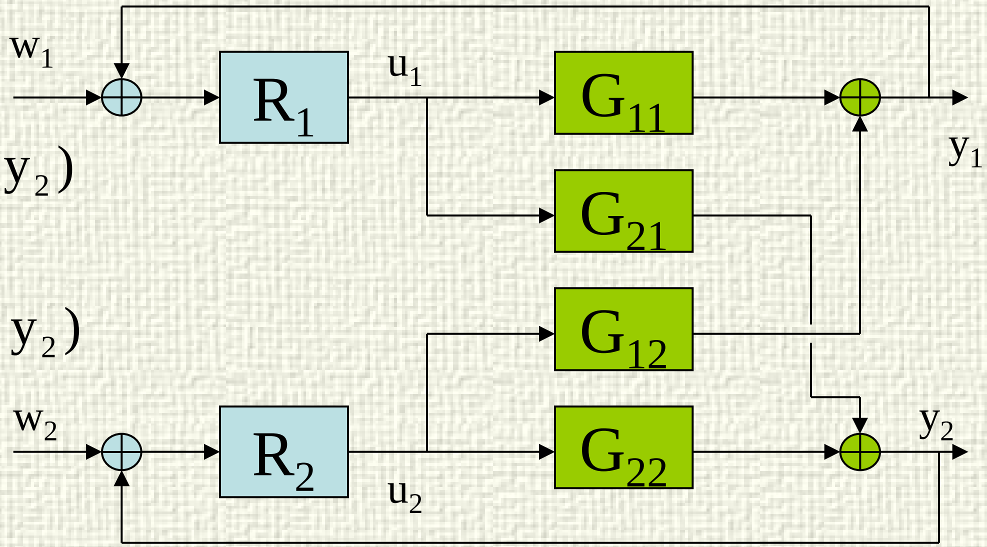
Interaction



Control with SISO controllers R

Closed loop

$$y_1 = G_{11}u_1 + G_{12}u_2 =$$
$$= G_{11}R_1(w_1 - y_1) + G_{12}R_2(w_2 - y_2)$$
$$y_2 = G_{21}u_1 + G_{22}u_2 =$$
$$= G_{21}R_1(w_1 - y_1) + G_{22}R_2(w_2 - y_2)$$

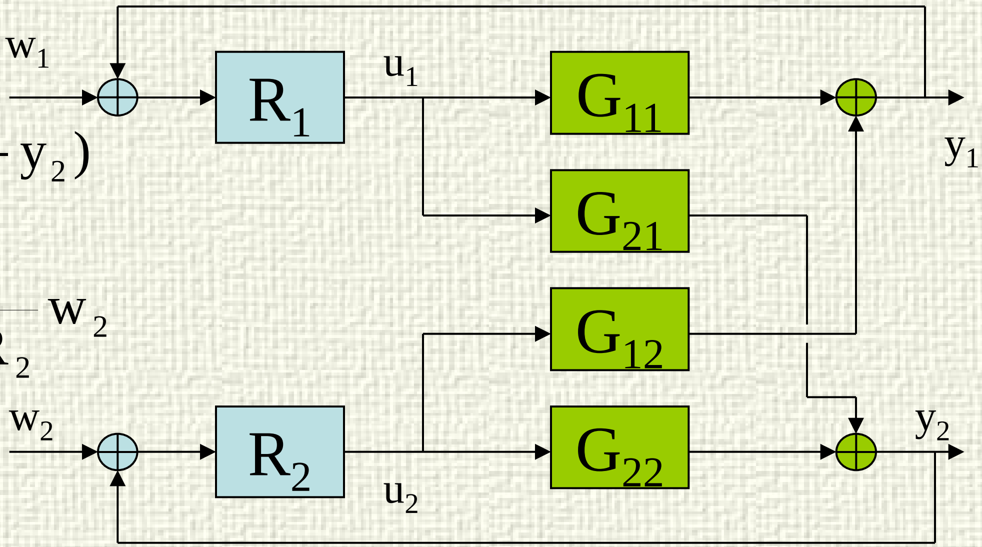


$$y_1 = \frac{G_{11}R_1}{1+G_{11}R_1}w_1 + \frac{G_{12}R_2}{1+G_{11}R_1}(w_2 - y_2)$$
$$y_2 = \frac{G_{21}R_1}{1+G_{22}R_2}(w_1 - y_1) + \frac{G_{22}R_2}{1+G_{22}R_2}w_2$$

Interaction

$$y_1 = \frac{G_{11}R_1}{1+G_{11}R_1} w_1 + \frac{G_{12}R_2}{1+G_{11}R_1}(w_2 - y_2)$$

$$y_2 = \frac{G_{21}R_1}{1+G_{22}R_2}(w_1 - y_1) + \frac{G_{22}R_2}{1+G_{22}R_2} w_2$$



$$y_1 = \frac{G_{11}R_1}{1+G_{11}R_1} w_1 + \frac{G_{12}R_2}{1+G_{11}R_1}\left(w_2 - \frac{G_{21}R_1}{1+G_{22}R_2}(w_1 - y_1) - \frac{G_{22}R_2}{1+G_{22}R_2} w_2\right)$$

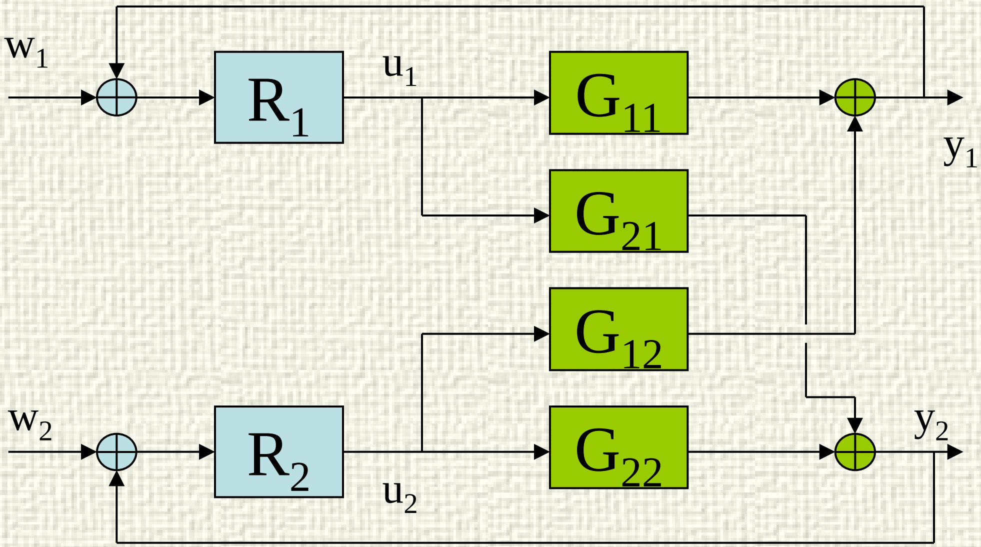
$$y_1 = \frac{G_{11}R_1(1+G_{22}R_2) - G_{12}R_2G_{21}R_1}{(1+G_{11}R_1)(1+G_{22}R_2) - G_{12}R_2G_{21}R_1} w_1 + \frac{G_{12}R_2}{(1+G_{11}R_1)(1+G_{22}R_2) - G_{12}R_2G_{21}R_1} w_2$$

Interaction (Loop 1)

w_1 y w_2 affect y_1

If G_{12} or G_{21} are = 0 the closed loop dynamics is the one of a SISO system u_1 --- y_1

If R_2 is commuted to manual the dynamics of loop 1 changes

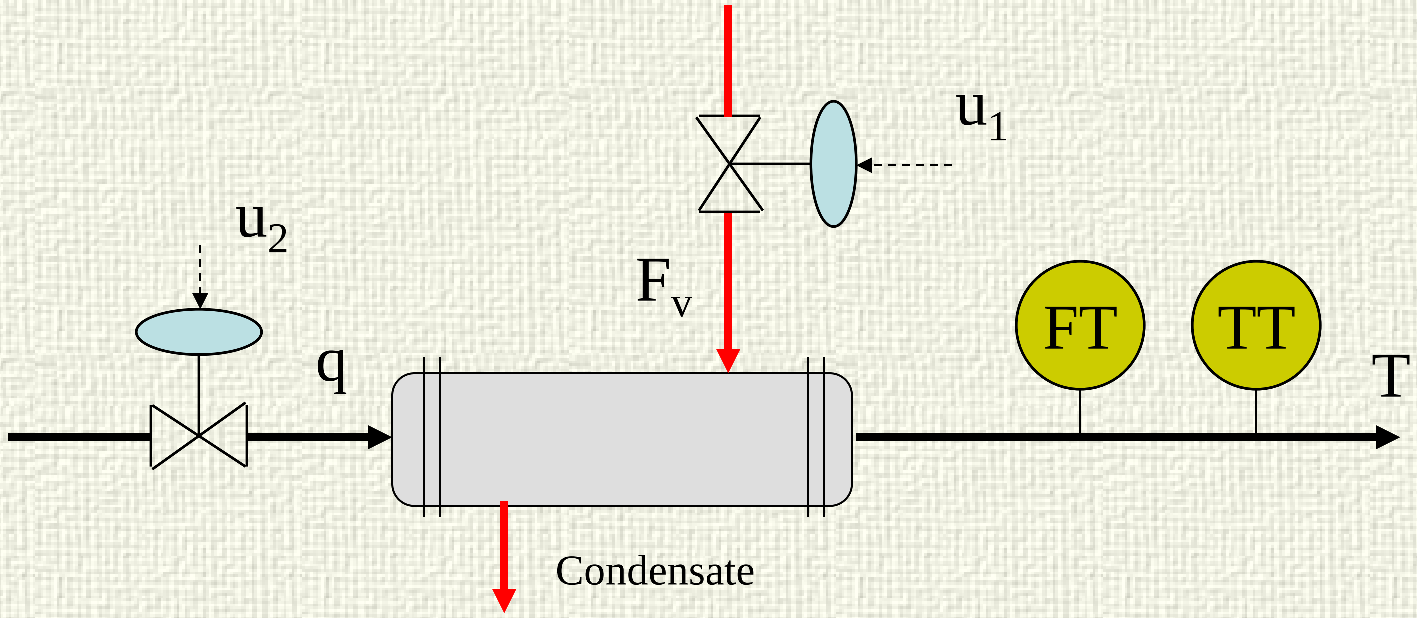


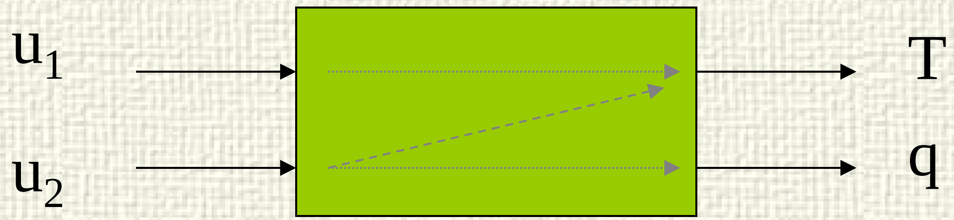
$$y_1 = \frac{G_{11}R_1(1+G_{22}R_2) - G_{12}R_2G_{21}R_1}{(1+G_{11}R_1)(1+G_{22}R_2) - G_{12}R_2G_{21}R_1} w_1 + \frac{G_{12}R_2}{(1+G_{11}R_1)(1+G_{22}R_2) - G_{12}R_2G_{21}R_1} w_2$$

$$\Downarrow$$

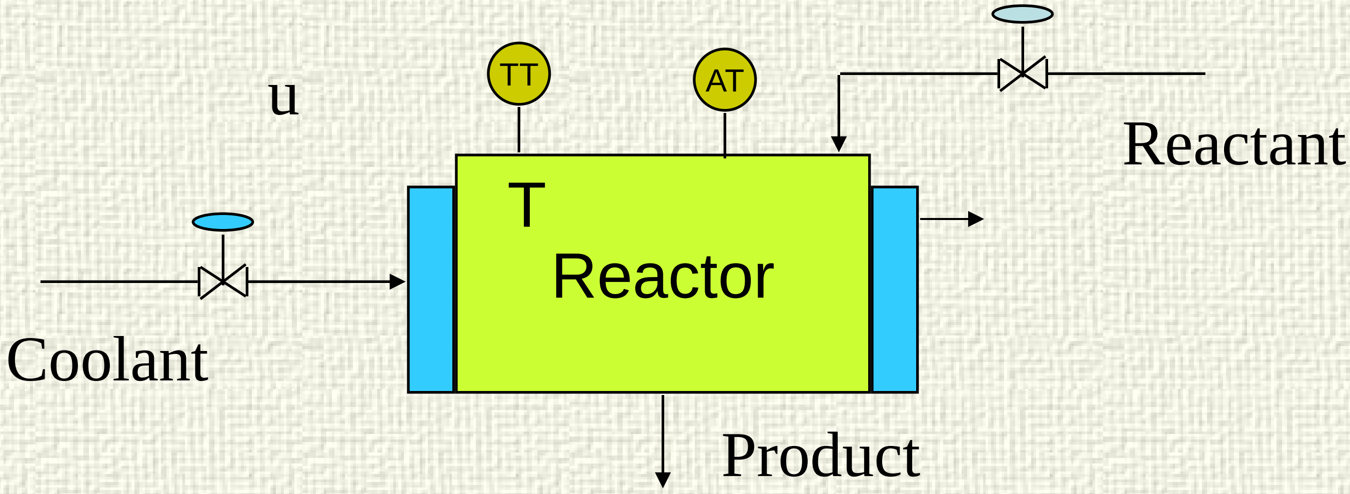
$$y_1 = \frac{G_{11}R_1}{(1+G_{11}R_1)} w_1 + \frac{G_{12}R_2}{(1+G_{11}R_1)(1+G_{22}R_2)} w_2 \qquad y_1 = \frac{G_{11}R_1}{(1+G_{11}R_1)} w_1$$

Interaction





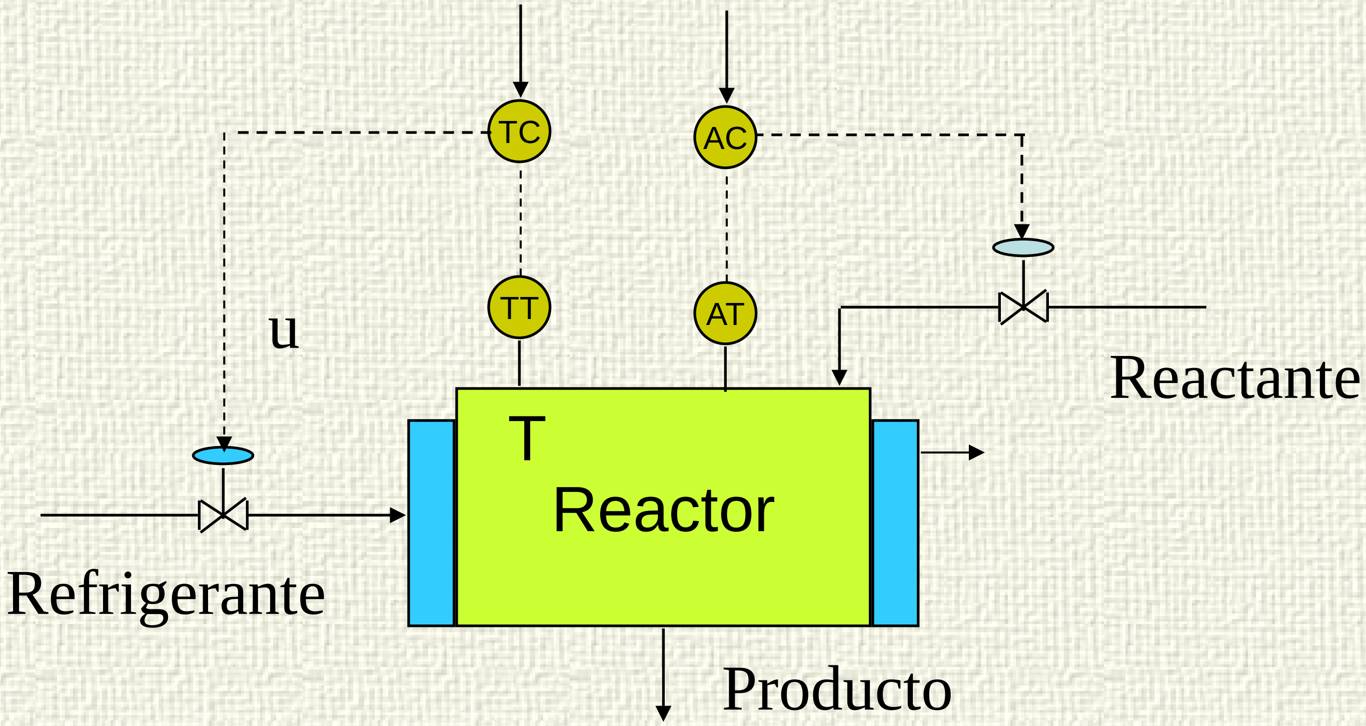
Reactor



Input output interaction in both variables

Open loop interaction

Reactor



Input - output interaction in both variables

Closed loop interaction

Interaction

- How to measure the degree of interaction?
- Is is possible to control the process using SISO controllers?
- If so, which is the best pairing of input – output variables?

Steady state gain matrix

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

The steady state gain matrix is not a good measure of interaction:

✓It depends on the units of the different variables

✓It does not reflect the main characteristic associated to interaction: the change in gain in a control loop when other loop switch from auto to man or vice versa.

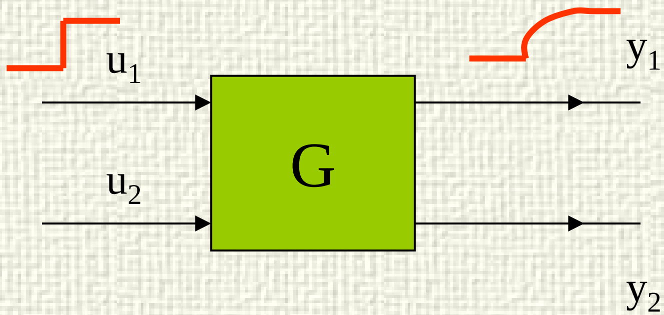
Bristol Relative Gain Array (RGA)

Bristol 1966 McAvoy 1983

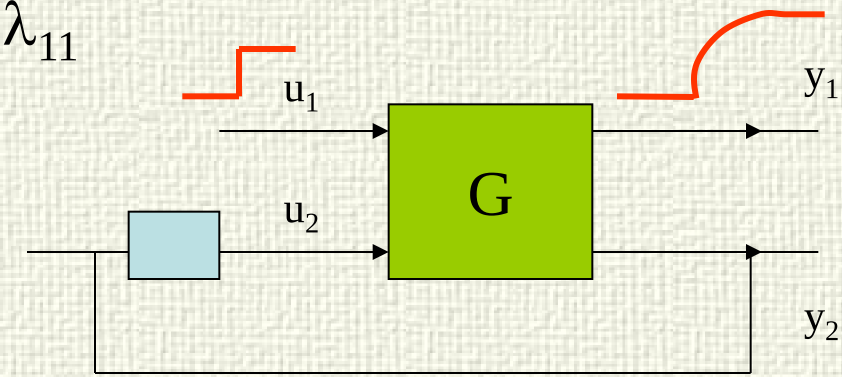
	u_1	u_2
y_1	λ_{11}	λ_{12}
y_2	λ_{21}	λ_{22}

$$\lambda_{i,j} = \frac{\left.\dfrac{\partial y_i}{\partial u_j}\right|_{u_k = cte; \forall k \neq j}}{\left.\dfrac{\partial y_i}{\partial u_j}\right|_{y_m = cte; \forall m \neq i}}$$

λ_{11} measures the change in gain between u_1 and y_1 in the experiments described below



λ_{11}



RGA

The RGA can be used to choose adequately the pairing of manipulated and controlled variables in MIMO systems, selecting those pairs with minimum interaction in steady state (or at any other frequency).

$$\lambda_{i,j} = \frac{\left.\dfrac{\partial y_i}{\partial u_j}\right|_{u_k = cte; \forall k \neq j}}{\left.\dfrac{\partial y_i}{\partial u_j}\right|_{y_m = cte; \forall m \neq i}}$$

$\lambda_{i,j} = 1$ ⇒ Best choice

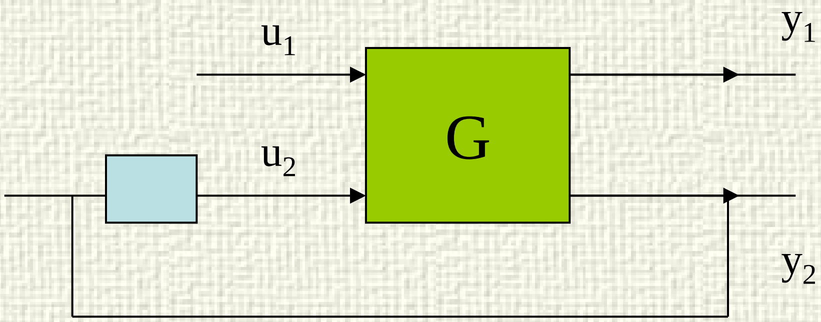
$\lambda_{i,j} = 0$

$\lambda_{i,j} = \infty$

	u_1	u_2
y_1	0.2	0.8
y_2	0.8	0.2

$\lambda_{i,j} < 0$ ⇒ Instability

RGA



$$\lambda_{11}$$

$$\Delta y_1 = k_{11}\Delta u_1 + k_{12}\Delta u_2$$
$$\Delta y_2 = 0 = k_{21}\Delta u_1 + k_{22}\Delta u_2$$

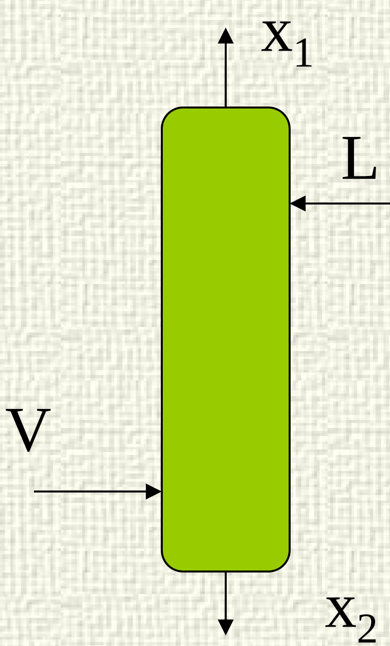
$$\Delta y_1 = k_{11}\Delta u_1 - \frac{k_{12}k_{21}}{k_{22}}\Delta u_1$$

$$\left.\frac{\Delta y_1}{\Delta u_1}\right|_{y_2=cte} = \frac{k_{11}k_{22} - k_{12}k_{21}}{k_{22}}$$

$$\lambda_{11} = \frac{k_{11}}{\frac{k_{11}k_{22} - k_{12}k_{21}}{k_{22}}} = \frac{k_{11}k_{22}}{k_{11}k_{22} - k_{12}k_{21}}$$

	u_1	u_2
y_1	$\frac{k_{11}k_{22}}{k_{11}k_{22} - k_{12}k_{21}}$	$\frac{-k_{12}k_{21}}{k_{11}k_{22} - k_{12}k_{21}}$
y_2	$\frac{-k_{12}k_{21}}{k_{11}k_{22} - k_{12}k_{21}}$	$\frac{k_{11}k_{22}}{k_{11}k_{22} - k_{12}k_{21}}$

Example: Distillation Column



$$\begin{matrix} & \% & \% & \\ G(0) = & \begin{bmatrix} 0.99 & -0.82 \\ 0.38 & -0.35 \end{bmatrix} & & \begin{matrix} \% \\ \% \end{matrix} \end{matrix}$$

$$\text{RGA} \quad \begin{matrix} & L & V \\ x_1 & 9{,}9 & -8{,}9 \\ x_2 & -8{,}9 & 9.9 \end{matrix}$$

Strong interaction associated to the pairing (L x_1) (V x_2)

Instability with (L x_2) (V x_1)

RGA

$$\mathrm{RGA}(G) = \Lambda(G) = G \times (G^{-1})^T$$

$$G = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$$

$$G^{-1} = \begin{bmatrix} 0.4 & 0.2 \\ -0.3 & 0.1 \end{bmatrix}$$

$$\Lambda(G) = G \times (G^{-1})^T = \begin{bmatrix} 0.4 & 0.6 \\ 0.6 & 0.4 \end{bmatrix}$$

Sum of elements of a RGA row or column is 1

RGA does not depend on the units or scaling of u and y

When dealing with asymmetric processes, the inverse matrix can be substituted by the pseudoinverse

Matlab RGA = G.*pinv(G)'

Example

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 4 & 0.5 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \qquad \text{pinv}(G) = \begin{bmatrix} 0.46 & -0.35 \\ -0.02 & 0.26 \\ 0.14 & -0.08 \end{bmatrix}$$

$$\Lambda(G) = G \times (G^{-1})^T = \begin{matrix} & u_1 & u_2 & u_3 \\ y_1 & 0.91 & -0.05 & 0.14 \\ y_2 & 0 & 1.04 & -0.04 \end{matrix}$$

y_1 must be paired with u_1

y_2 must be paired with u_2

Example RGA

$$G = \begin{bmatrix} 16.8 & 30.5 & 4.3 \\ -16.7 & 31.0 & -1.41 \\ 1.27 & 54.1 & 5.40 \end{bmatrix}$$

$$\begin{array}{c c} & \begin{array}{ccc} u_1 & u_2 & u_3 \end{array} \\ \begin{array}{c} y_1 \\ y_2 \\ y_3 \end{array} & \begin{bmatrix} 1.50 & 0.99 & -1.48 \\ -0.41 & 0.97 & 0.45 \\ -0.08 & -0.95 & 2.03 \end{bmatrix} \end{array}$$

RGA

The only admissible SISO pairing is:

y_1 ---- u_1 y_2 ---- u_2 y_3 ---- u_3

With a higher interaction in the third loop

RGA

$$G = \begin{bmatrix} 16.8 & 30.5 & 4.3 \\ -16.7 & 31.0 & -1.41 \\ 1.27 & 54.1 & 5.40 \end{bmatrix}$$

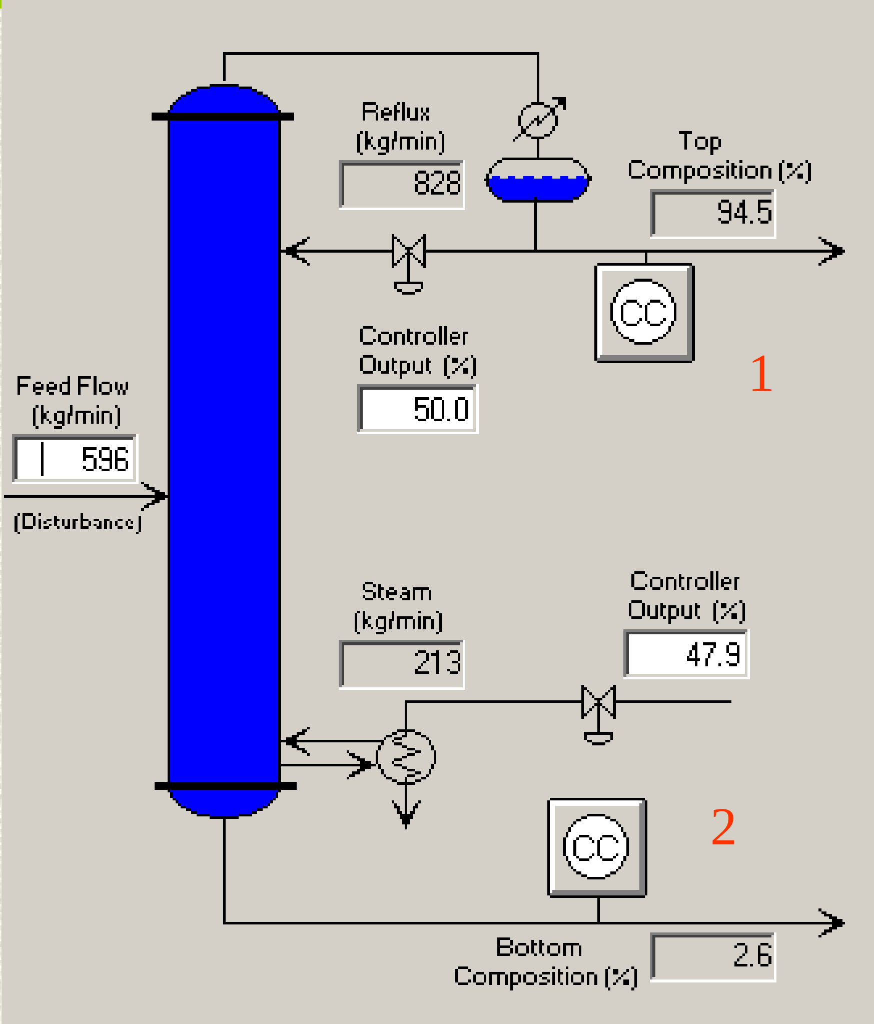
$$\begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \end{array} \begin{array}{c} \begin{array}{ccc} u_1 & u_2 & u_3 \end{array} \\ \begin{bmatrix} 1.50 & 0.99 & -1.48 \\ -0.41 & 0.97 & 0.45 \\ -0.08 & -0.95 & 2.03 \end{bmatrix} \end{array}$$

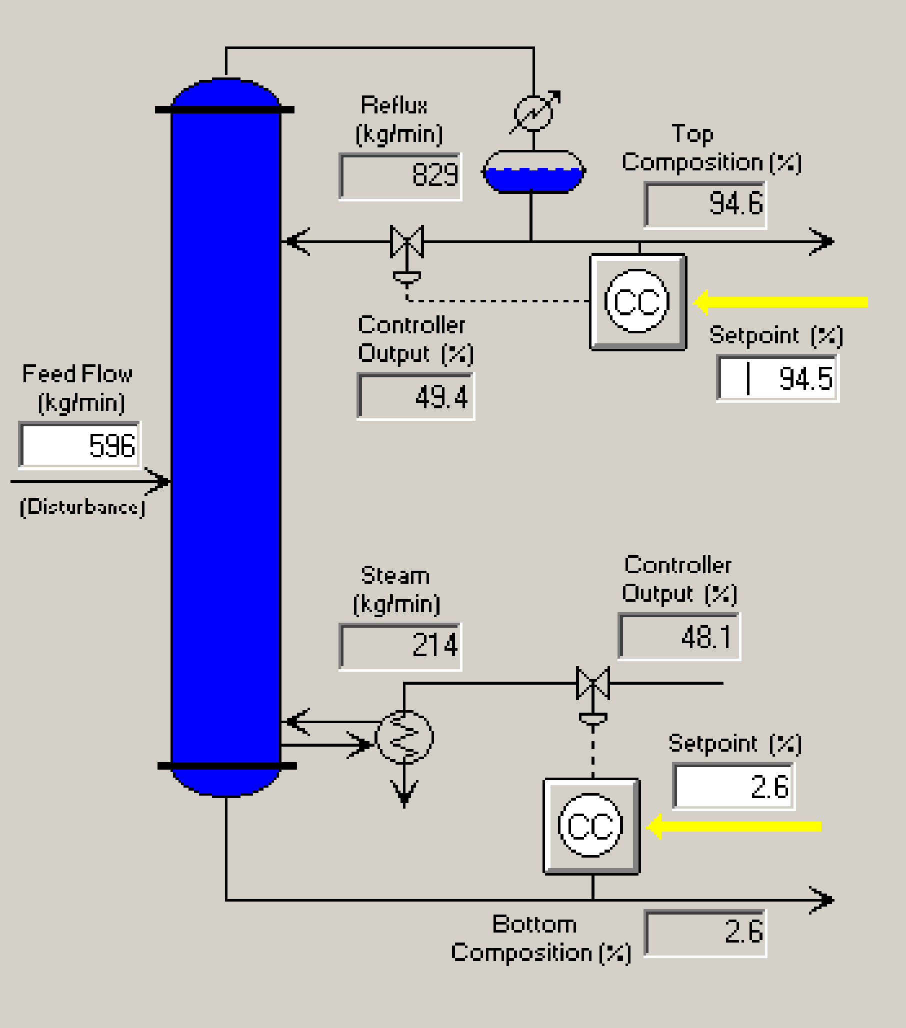
RGA

$$\lambda_{i,j} = \frac{\left.\dfrac{\partial y_i}{\partial u_j}\right|_{u_k = cte; \forall k \neq j}}{\left.\dfrac{\partial y_i}{\partial u_j}\right|_{y_m = cte; \forall m \neq i}}$$

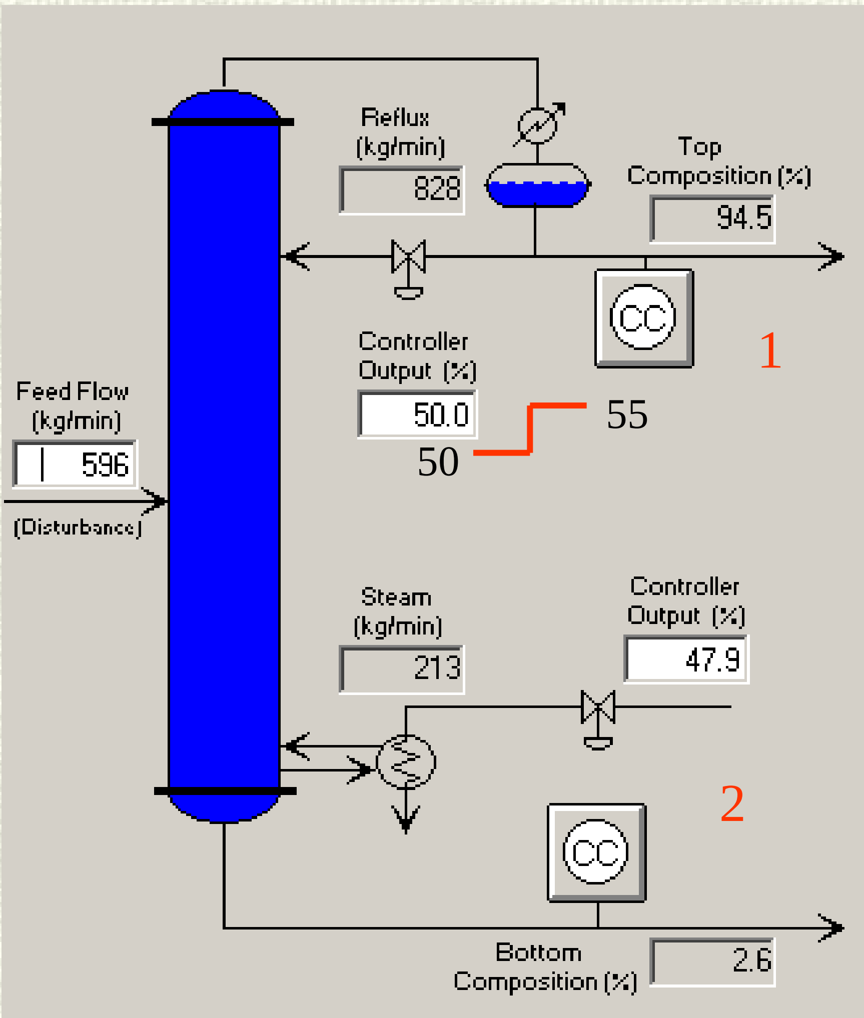
Notice that, if λ > 1, when changing from auto to man, the resulting gain will be larger than before and, likely, the loop will tend to oscillate. By the contrary, if λ < 1, will provide a slower response.

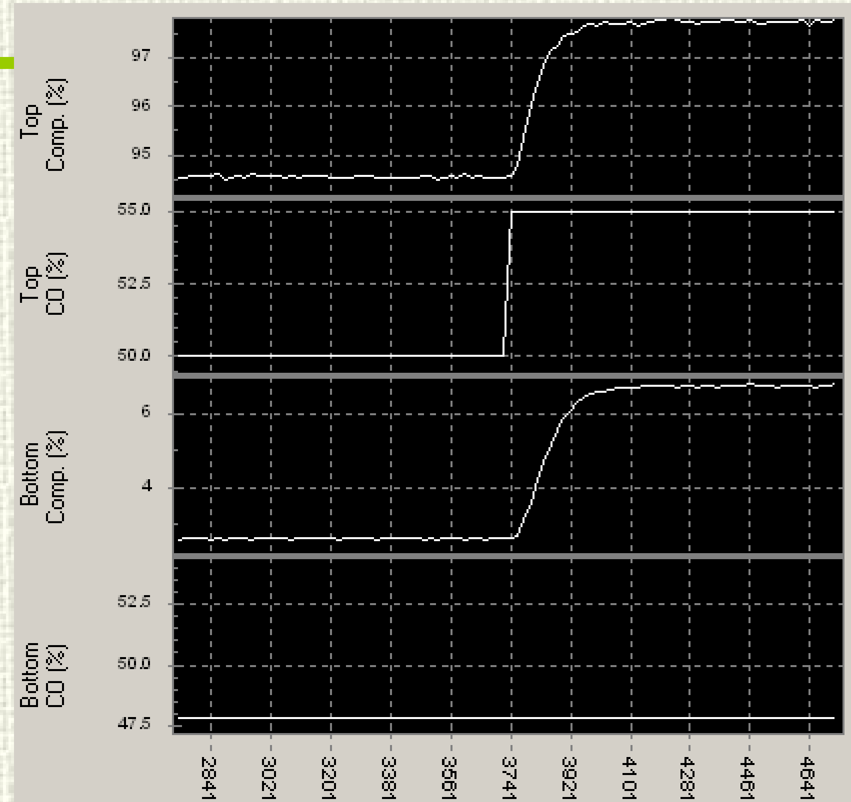
Distillation Column





Open loop experiment

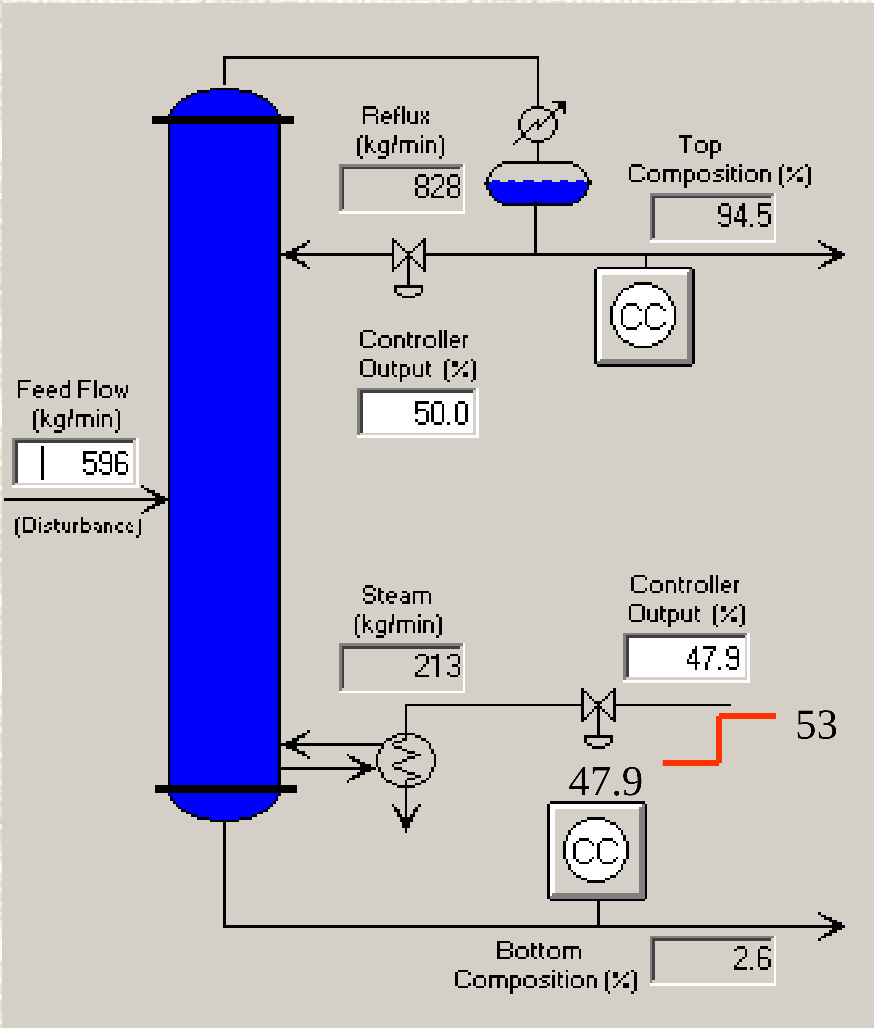


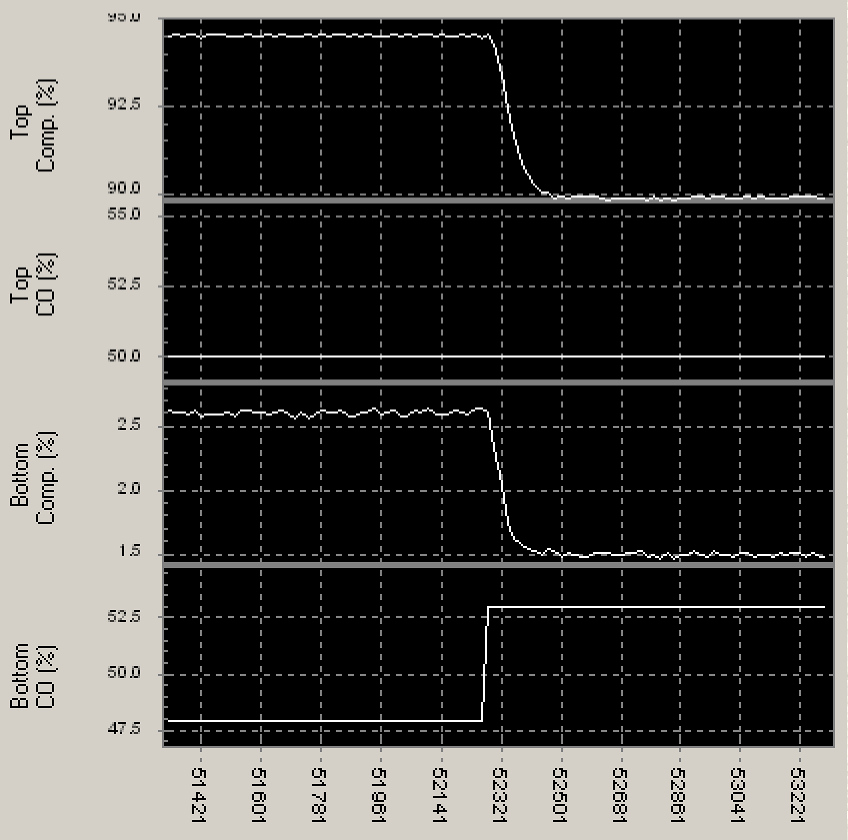


$$G_{11} = \frac{K_{11} e^{-d_{11}s}}{\tau_{11}s+1} = \frac{0.648e^{-21.7s}}{60s+1}$$

$$G_{21} = \frac{K_{21} e^{-d_{21}s}}{\tau_{21}s+1} = \frac{0.815e^{-34.4s}}{84.7s+1}$$

Open loop experiment

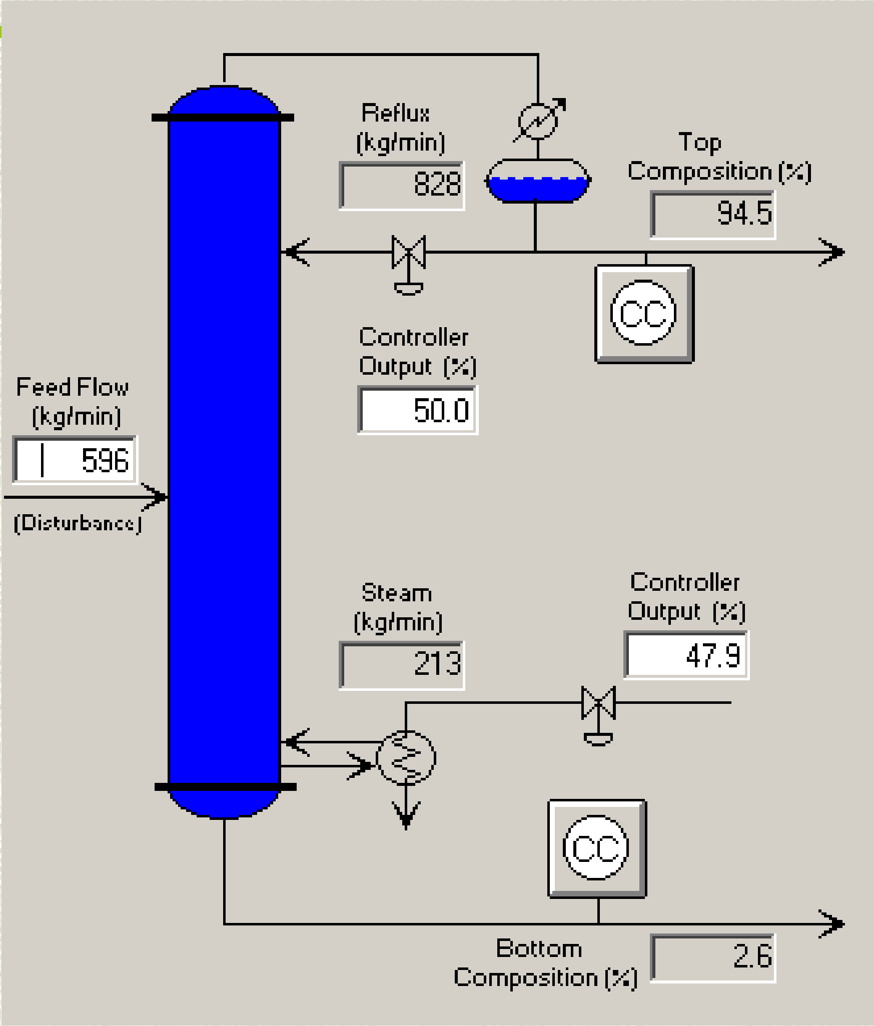




$$G_{12} = \frac{K_{12} e^{-d_{12} s}}{\tau_{12} s + 1} = \frac{-0.894 e^{-21.6s}}{54.3s + 1}$$

$$G_{22} = \frac{K_{22} e^{-d_{22} s}}{\tau_{22} s + 1} = \frac{-0.236 e^{-6.61s}}{41.9s + 1}$$

Open loop Model



FOPD Step response models

Both loops open

$$G_{11} = \frac{K_{11}e^{-d_{11}s}}{\tau_{11}s+1} = \frac{0.648e^{-21.7s}}{60s+1}$$

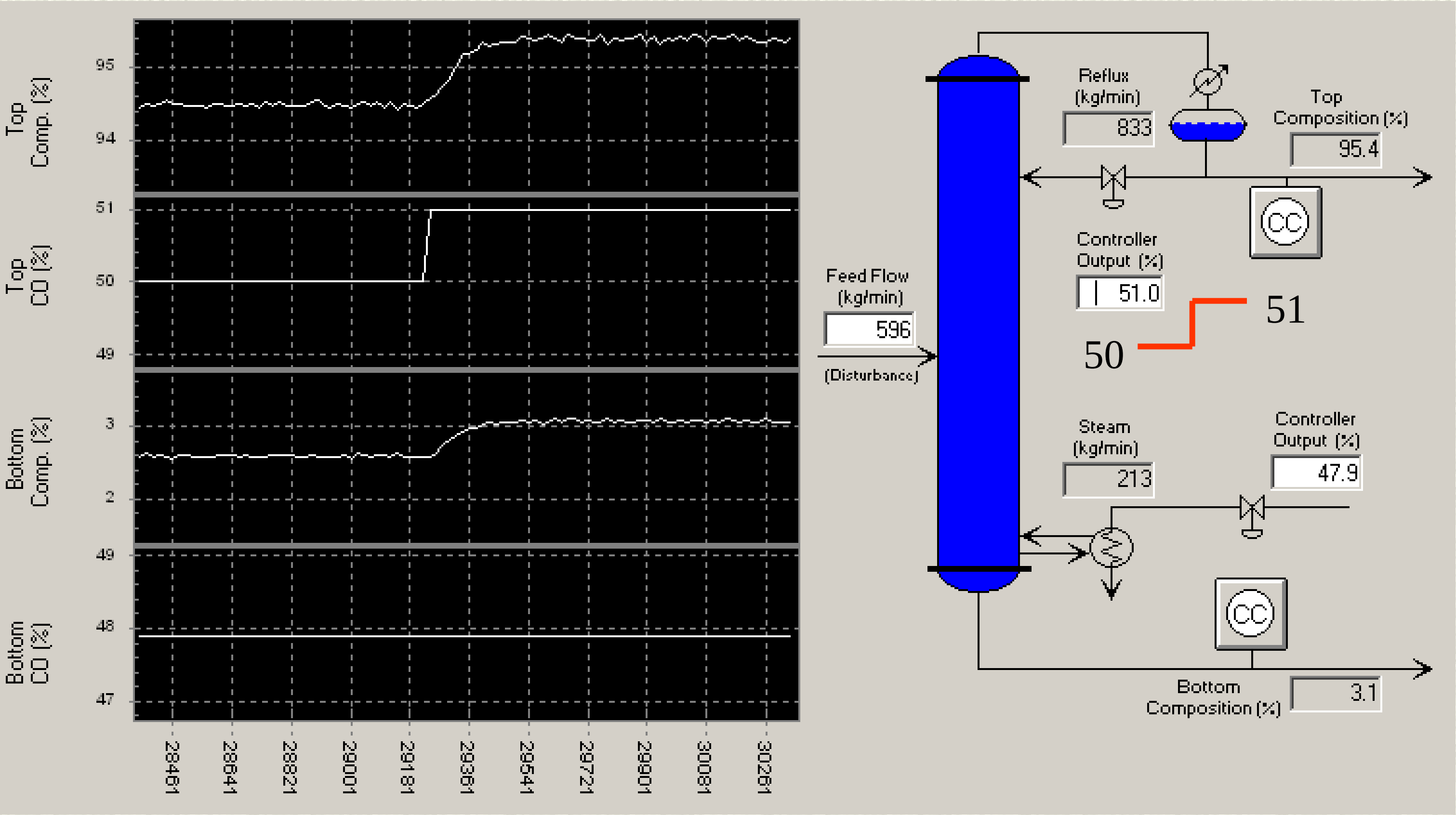
$$G_{21} = \frac{K_{21}e^{-d_{21}s}}{\tau_{21}s+1} = \frac{0.815e^{-34.4s}}{84.7s+1}$$

$$G_{12} = \frac{K_{12}e^{-d_{12}s}}{\tau_{12}s+1} = \frac{-0.894e^{-21.6s}}{54.3s+1}$$

$$G_{22} = \frac{K_{22}e^{-d_{22}s}}{\tau_{22}s+1} = \frac{-0.236e^{-6.61s}}{41.9s+1}$$

Watch the experiment!

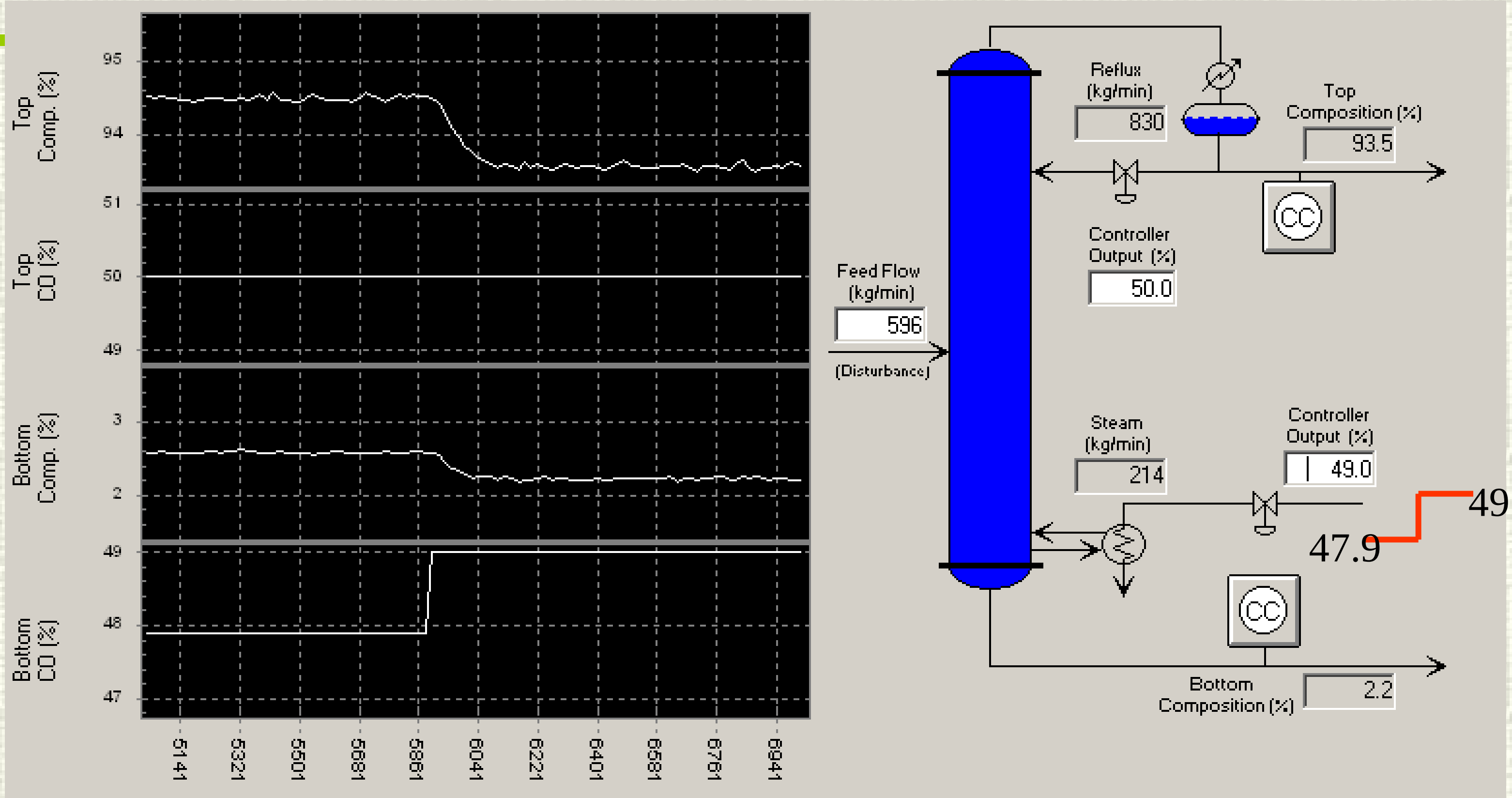
Change of 0.5%



In non-linear systems, smaller input changes can provide better results

$$G_{11} = \frac{K_{11}e^{-d_{11}s}}{\tau_{11}s+1} = \frac{0.99e^{-22.7s}}{72.8s+1} \qquad G_{21} = \frac{K_{21}e^{-d_{21}s}}{\tau_{21}s+1} = \frac{0.38e^{-30.9s}}{66.65s+1}$$

Plan well the experiment



$$G_{12} = \frac{K_{12}e^{-d_{12}s}}{\tau_{12}s+1} = \frac{-0.82e^{-22.36s}}{66.67s+1} \qquad G_{22} = \frac{K_{22}e^{-d_{22}s}}{\tau_{22}s+1} = \frac{-0.35e^{-4.5s}}{57.02s+1}$$

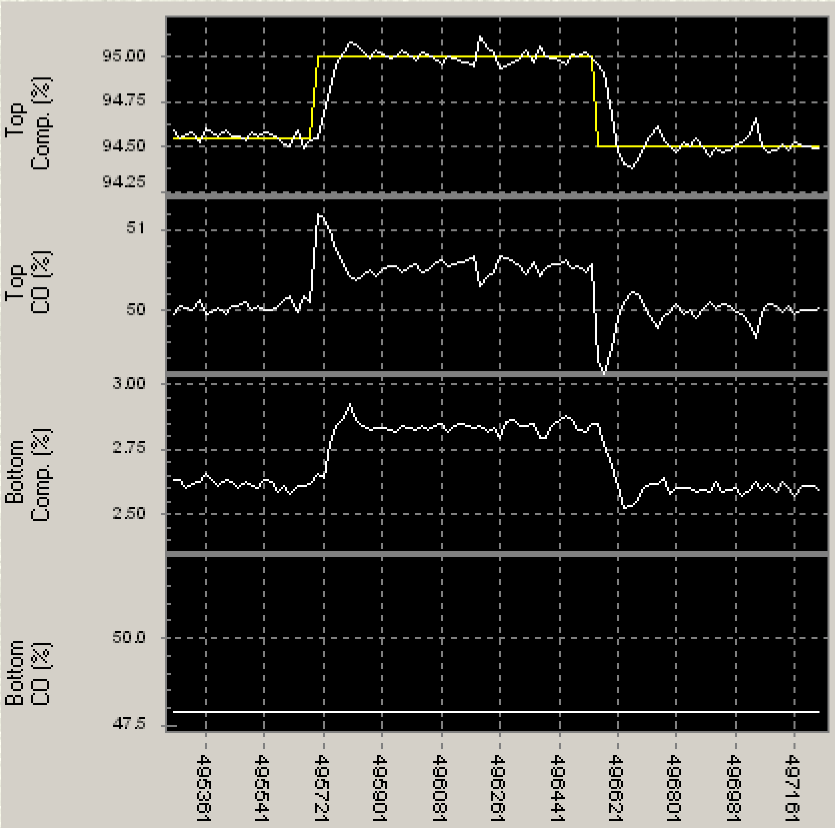
$$K = \begin{bmatrix} 0.648 & -0.894 \\ 0.815 & -0.236 \end{bmatrix}$$

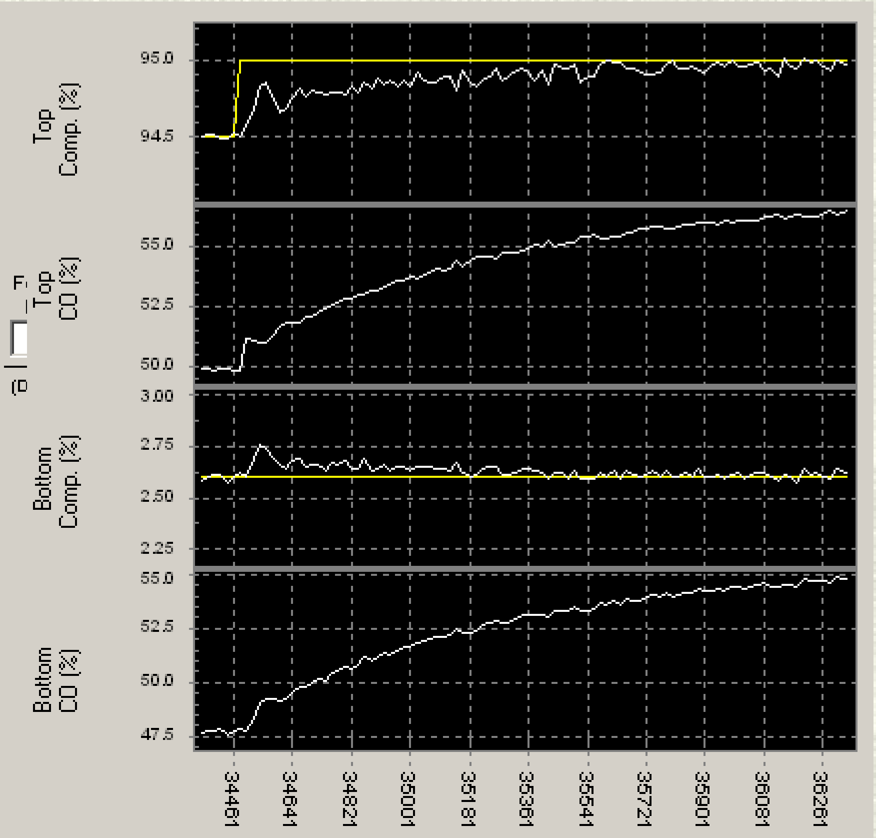
RGA

$$RGA = \begin{bmatrix} -0.265 & 1.265 \\ 1.265 & -0.265 \end{bmatrix}$$

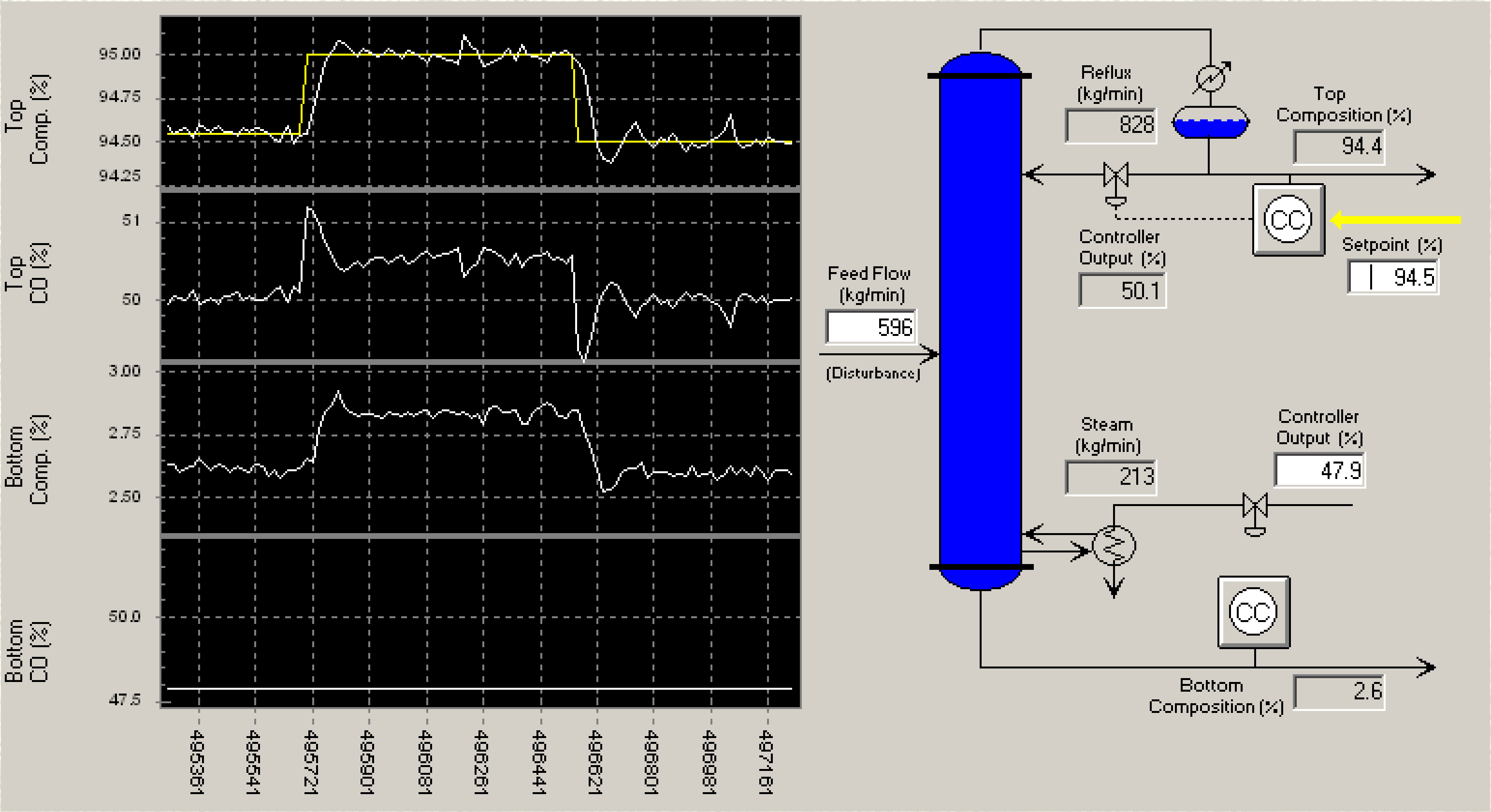
$$K = \begin{bmatrix} 0.99 & -0.82 \\ 0.38 & -0.35 \end{bmatrix}$$

$$RGA = \begin{bmatrix} 9.1 & -8.1 \\ -8.1 & 9.1 \end{bmatrix}$$



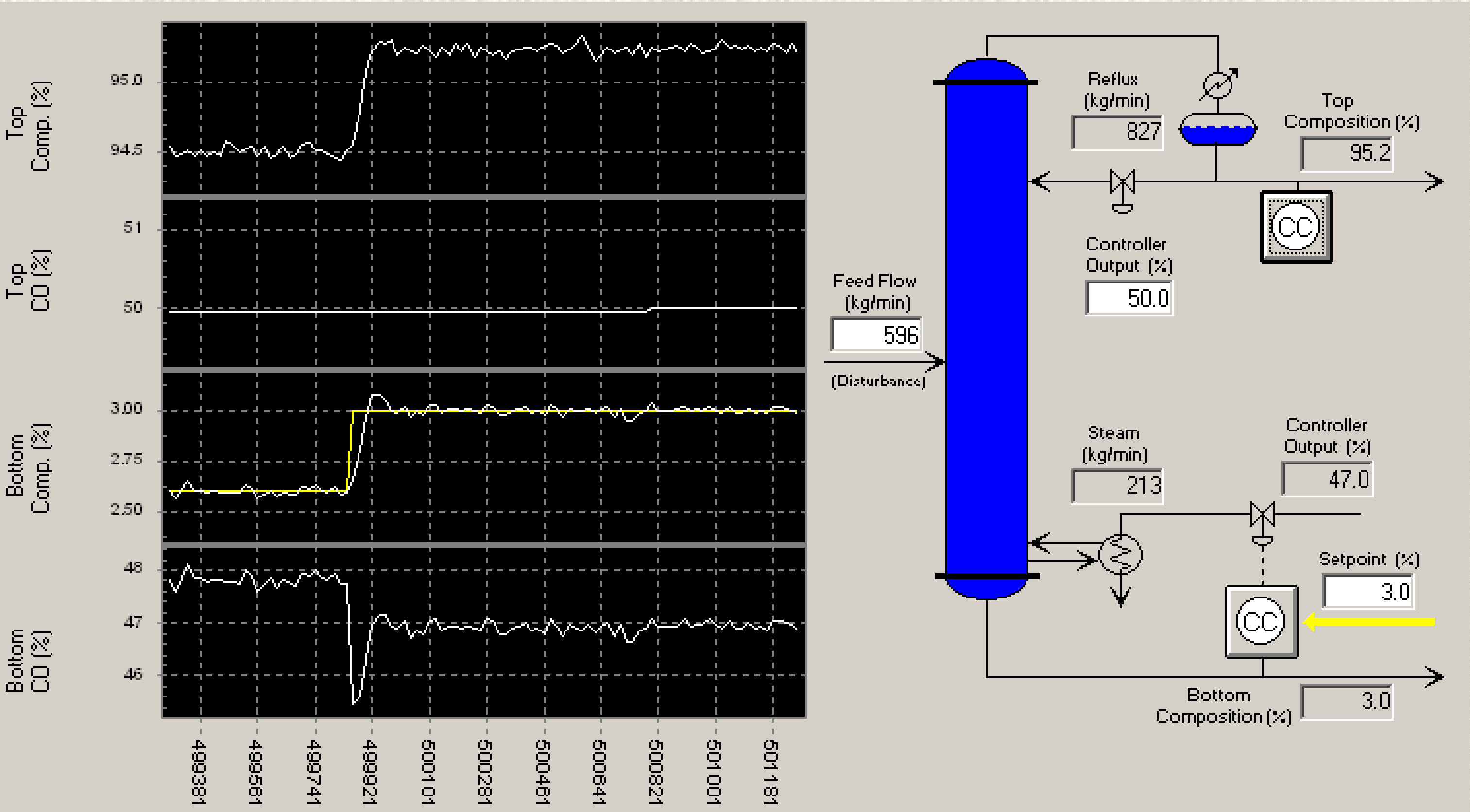


Head composition control with bottom impurity control in manual



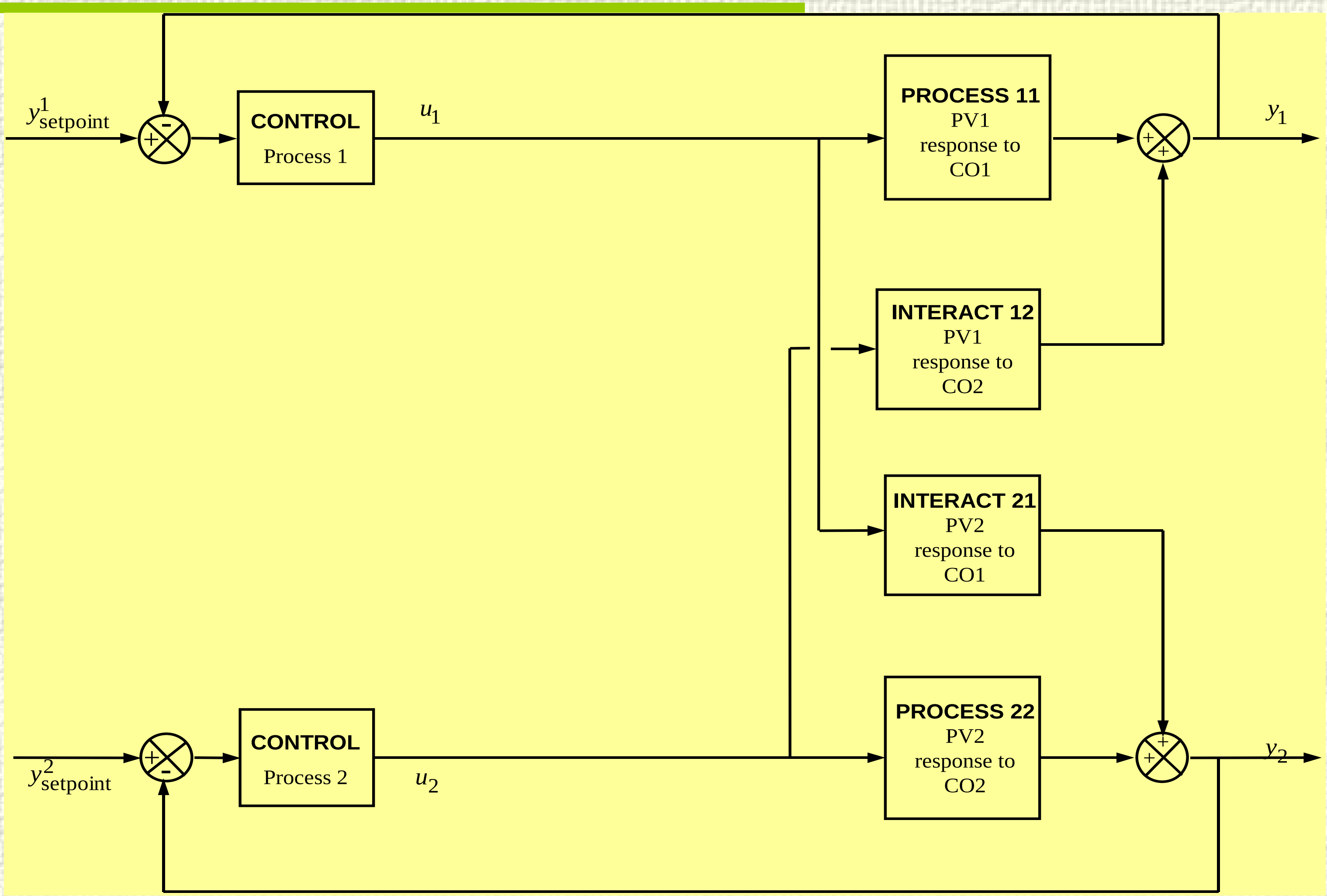
IMC tuning λ=50 min. K_p = 2.19, T_i = 70.85 min.

Bottom impurity control with head composition control in manual



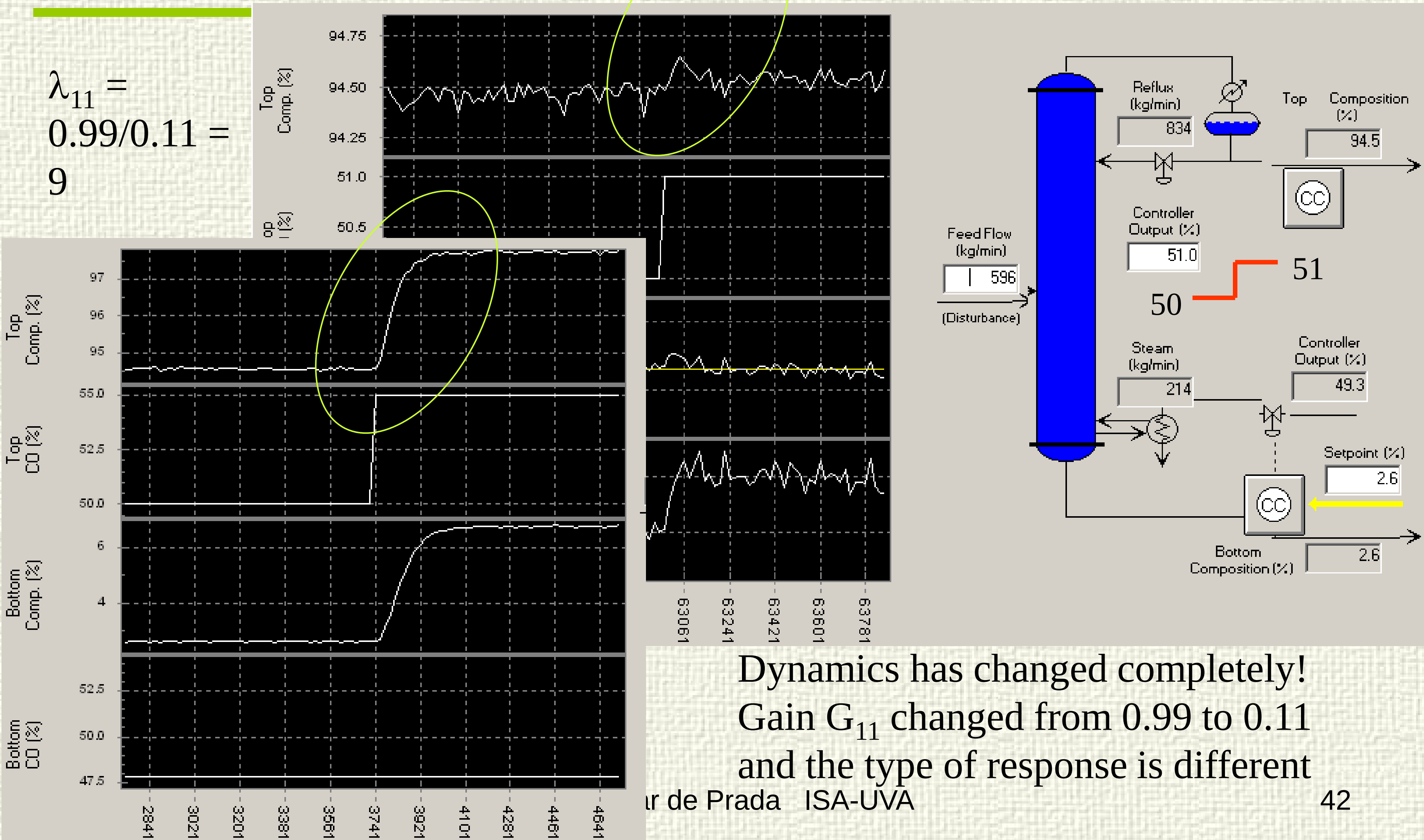
IMC tuning λ=35 min. K_p = -5.47, T_i = 45.2 min.

Control loop interaction



Open loop response G between 1-1 with the bottom impurity control in automatic

$\lambda_{11} = 0.99/0.11 = 9$



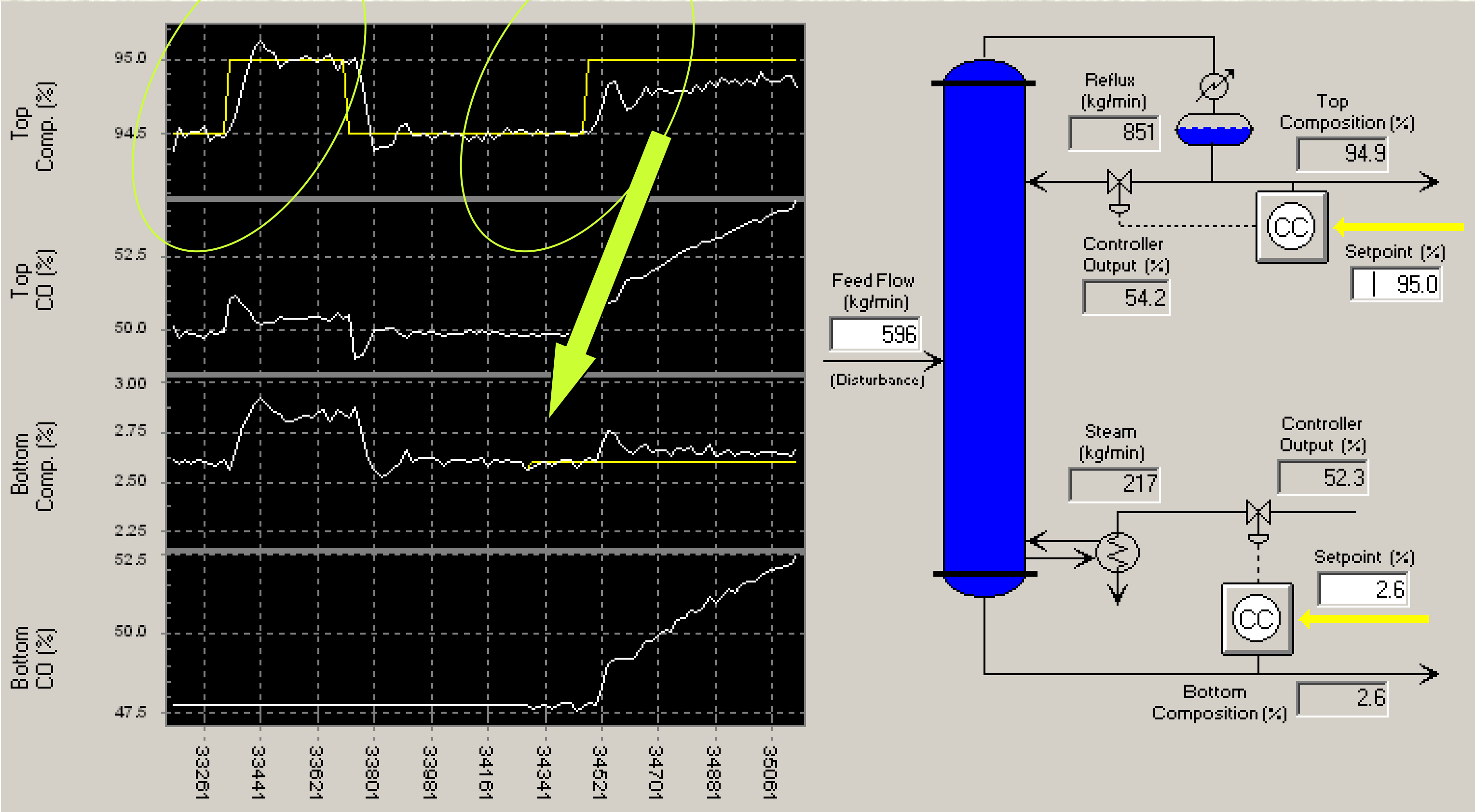
Dynamics has changed completely!

Gain G_{11} changed from 0.99 to 0.11 and the type of response is different

Head composition control with bottom impurity control in automatic

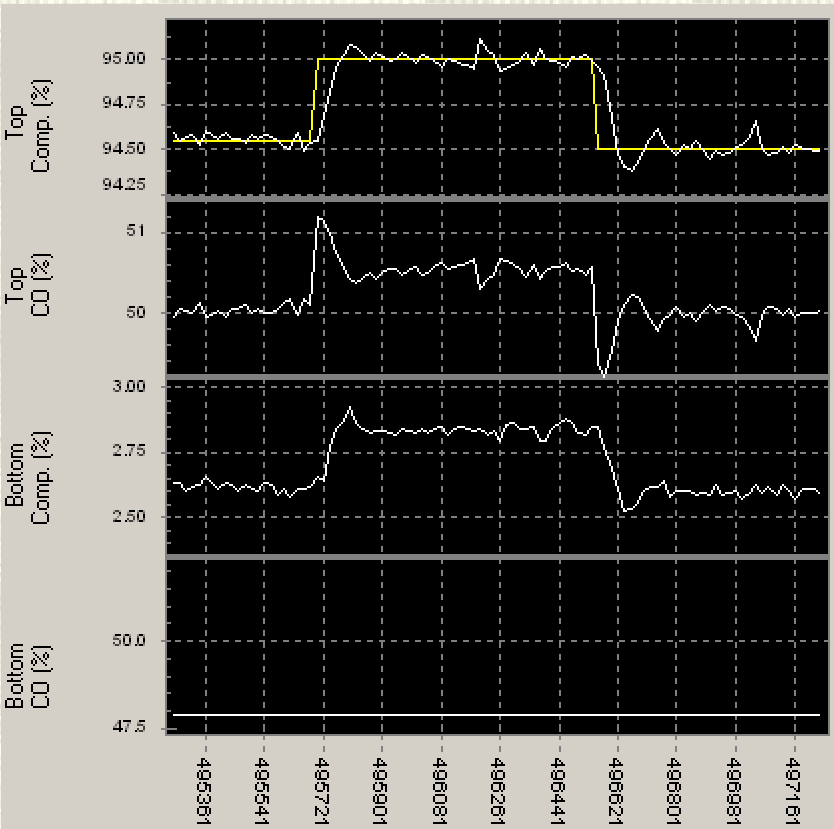
Switch from man to auto in the bottom impurity control loop

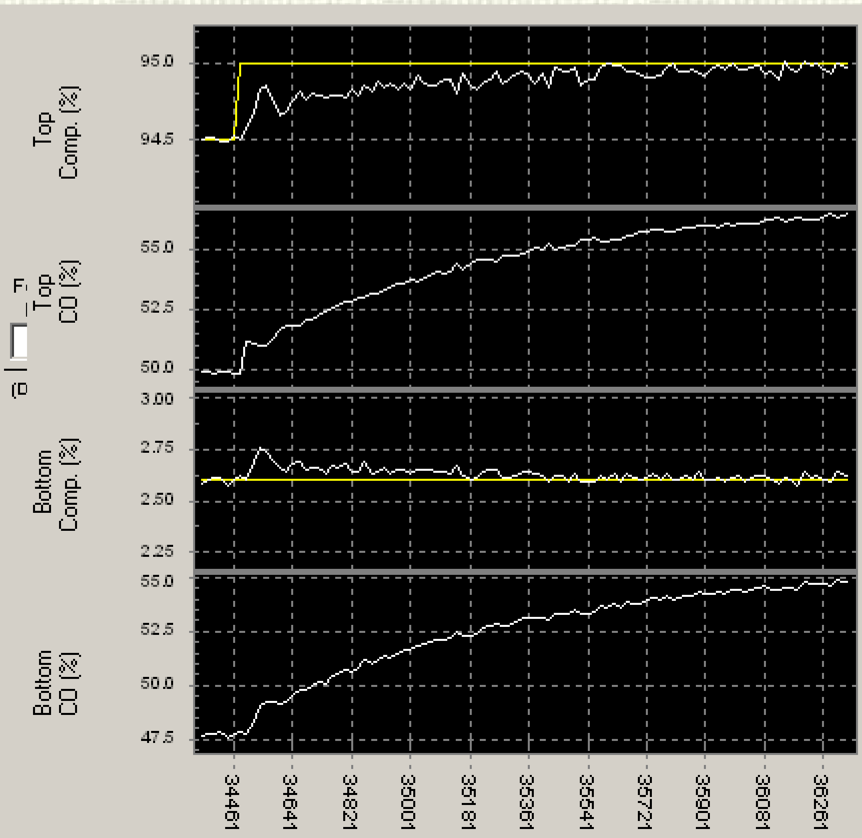
Process dynamics changes completely



RGA

$$\text{RGA} = \begin{bmatrix} 9.1 & -8.1 \\ -8.1 & 9.1 \end{bmatrix}$$



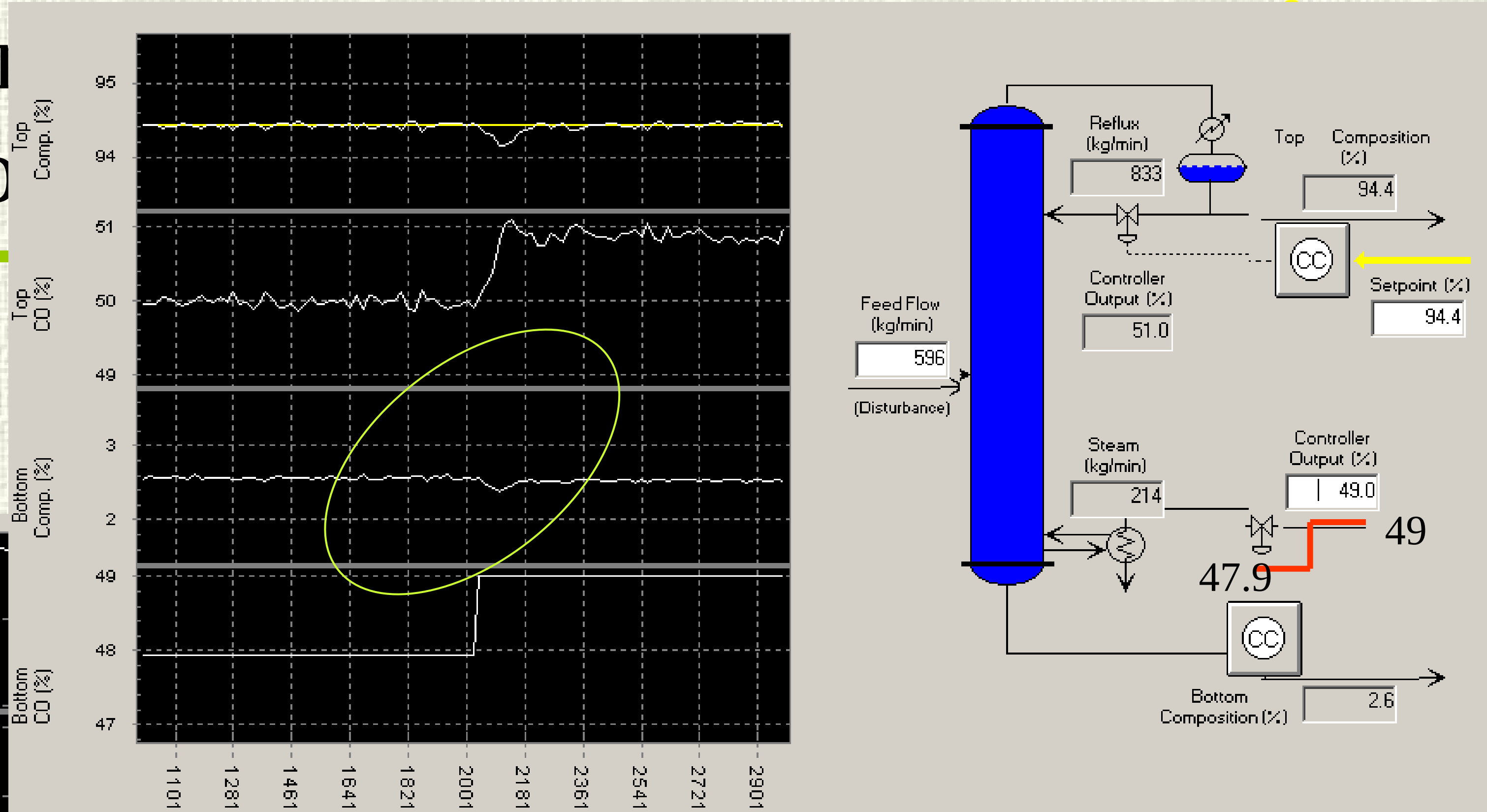


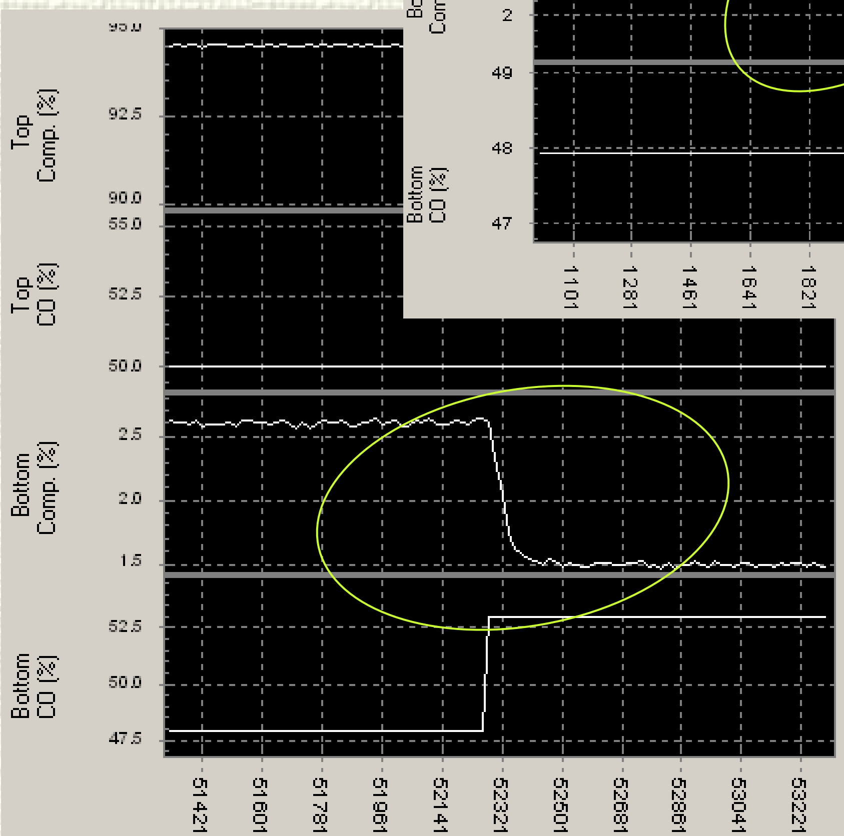
Ope
co

λ_{22} =

-0.35/(-0.043)

= 8.1



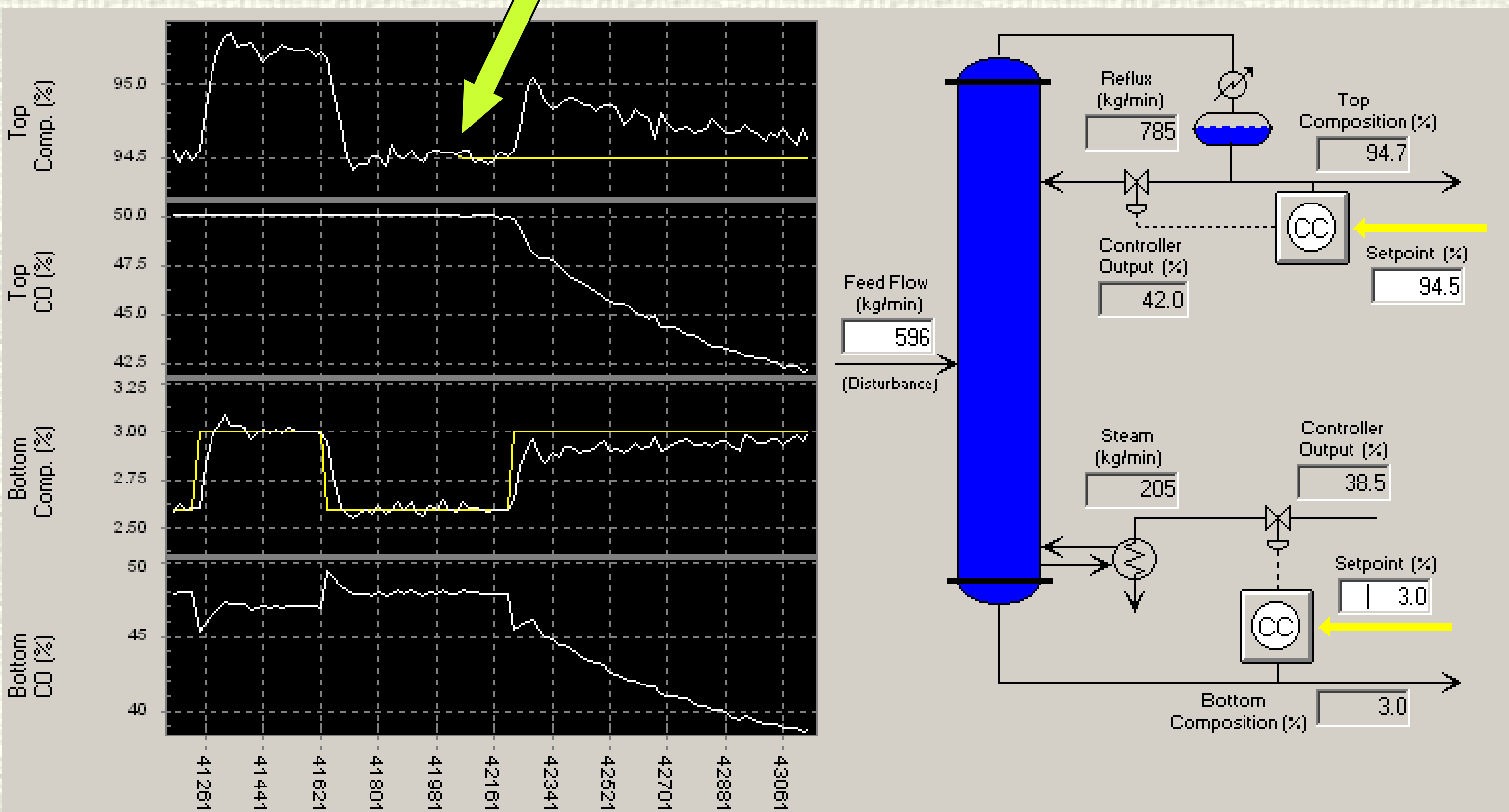


Dynamics has changed completely! Gain G_{22} changed from -0.35 to -0.043and the type of response is different

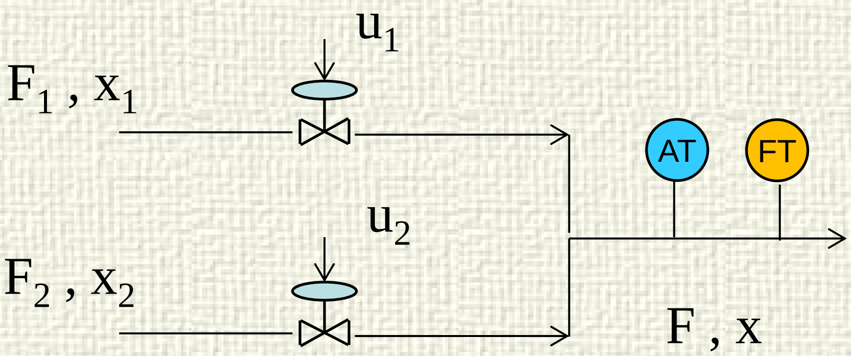
Bottom impurity control with head composition control in automatic

Switch from man to auto in the head composition control loop

Dynamics has changed completely



Example: Mixing two streams



Global balance:

$$F = F_1 + F_2$$

Composition balance:

$$F\,x = F_1\,x_1 + F_2\,x_2$$

$$x = \frac{F_1\,x_1 + F_2\,x_2}{F_1 + F_2} \qquad \Delta F = \Delta F_1 + \Delta F_2$$

$$\Delta x = \frac{(F_1 + F_2)x_1 - (F_1\,x_1 + F_2\,x_2)}{(F_1 + F_2)^2}\Delta F_1 + \frac{(F_1 + F_2)x_2 - (F_1\,x_1 + F_2\,x_2)}{(F_1 + F_2)^2}\Delta F_2$$

$$\Delta x = \left.\frac{F_2\,(x_1 - x_2)}{(F_1 + F_2)^2}\right|_{ss}\Delta F_1 - \left.\frac{F_1\,(x_1 - x_2)}{(F_1 + F_2)^2}\right|_{ss}\Delta F_2$$

Linearization in a certain steady state point

Example: Mixing two streams

Steady state gain matrix

$$\begin{bmatrix}\Delta x\\ \Delta F\end{bmatrix}=\begin{bmatrix}\left.\dfrac{F_2(x_1-x_2)}{(F_1+F_2)^2}\right|_{ss} & -\left.\dfrac{F_1(x_1-x_2)}{(F_1+F_2)^2}\right|_{ss}\\ 1 & 1\end{bmatrix}\begin{bmatrix}\Delta F_1\\ \Delta F_2\end{bmatrix}$$

$$F=F_1+F_2 \qquad F\,x=F_1\,x_1+F_2\,x_2$$

Eliminating F_2 between both equations:

$$F=F_1\cdot\frac{x_1-x_2}{x-x_2} \quad \Rightarrow$$

$$\lambda_{F,F_1}=\frac{1}{\left(\dfrac{x_1-x_2}{x-x_2}\right)}=\frac{x-x_2}{x_1-x_2}=\frac{F_1}{F}$$

$$Fx-Fx_2=F_1x_1-F_1x_2 \quad\Rightarrow Fx=F_1x_1-F_1x_2+Fx_2=F_1x_1-F_1x_2+F_1x_2+F_2x_2=F_1x_1+F_2x_2$$

Mixing two streams

RGA :

	F_1	F_2
F	$\frac{F_1}{F}$	$1-\frac{F_1}{F}$
x	$1-\frac{F_1}{F}$	$\frac{F_1}{F}$

Which is the best pairing between manipulated and controlled variables? What influences the answer?

Mixing two streams

	F_1	F_2
F	$\frac{F_1}{F}$	$1-\frac{F_1}{F}$
x	$1-\frac{F_1}{F}$	$\frac{F_1}{F}$

Case $F_1 = 3$, $F = 15$

	F_1	F_2
F	0.2	0.8
x	0.8	0.2

Case $F_1 = 10$, $F = 15$

	F_1	F_2
F	0.67	0.33
x	0.33	0.67

RGA(jω)

$$\text{RGA}(G(j\omega))$$

RGA was originally formulated for the steady state case (zero frequency), but the same concept can be applied to the operation of the process at any other frequency to measure the interaction during transients.

Niederlinski stability theorem

Let's assume that inputs and outputs of a multivarible system have been ordered so that y_1 is controlled with u_1, y_2 is controlled with u_2, etc. and each pair is controlled with a regulator having integral action, then, the closed loop is unstable if:

$$\frac{\det(G(0))}{\prod_{i=1}^{n} G_{ii}(0)} < 0$$

Singular values

Eigenvalues of G: $|G - \lambda I| = 0$

Spectral radious: $\rho(G) = \max_i |\lambda_i(G)|$

How to compute the eigenvalues of a non-square matrix?

Singular values

$$\sigma_i(G) = +\sqrt{\lambda_i(G^*G)} = +\sqrt{\lambda_i(GG^*)}$$

if $G(l \times m)$,

the $k = \min(l, m)$ largest eigenvalues of G^*G y GG^* are selected

$$\min \sigma_i = \underline{\sigma}(G) \quad \max \sigma_i = \bar{\sigma}(G) \quad \text{condition number} = \frac{\bar{\sigma}(G)}{\underline{\sigma}(G)}$$

$$\underline{\sigma}(G) \le |\lambda_i(G)| \le \bar{\sigma}(G) \qquad \underline{\sigma}(G) = 1/\bar{\sigma}(G^{-1})$$

SVD

$$G = U\Sigma V^* = \begin{bmatrix} u_1 & u_2 & \dots & u_l \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \dots & 0 & 0 \\ 0 & 0 & \sigma_n & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1^* \\ v_2^* \\ \dots \\ v_m^* \end{bmatrix}$$

$U(l \times l),\ V(m \times m)$ unitary matrices (orthonormals)

$$U^{-1} = U^* \quad V^{-1} = V^* \qquad u_i^* u_j = \delta_{ij} \quad \|u_i\|_2 = 1 \quad v_i^* v_j = \delta_{ij} \quad \|v_i\|_2 = 1$$

Columns of U: u_i output singular vectors: unitary eigenvectors of GG^*

Columns of V: v_i input singular vectors: unitary eigenvectors of G^*G

SVD

One input in the direction v_i gives an output in the direction u_i

$$G = U\Sigma V^* \Rightarrow \quad GV = U\Sigma V^* V = U\Sigma$$

$$Gv_i = \sigma_i u_i$$

σ_i gives the gain of G in the direction v_i

As $\|v_i\|_2 = 1, \quad \|u_i\|_2 = 1$

$$\|\sigma_i u_i\|_2 = \sigma_i \|u_i\|_2 = \sigma_i(G) = \|Gv_i\|_2 = \frac{\|Gv_i\|_2}{\|v_i\|_2}$$

Directions computed using SVD are orthogonals

Interaction using SVD

- Opposite to the RGA, the SVD depend on the scaling of the matrix, so, in oreder to obtin sensibles results, the G matrix should be scaled to units used in the controller

$$G = U\Sigma V^*$$

- First column of V (row of V*) provides the combination of controller moves with the largest effect on the controlled variables, these ones changing in the direction given by the first column of U. The second column of V provides the second largest effect, etc.These gains are given by the singular values σ_i

Pairing variables with SVD

$$G = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}$$

$$G = U\Sigma V^* = \begin{bmatrix} -0.872 & -0.490 \\ -0.490 & 0.872 \end{bmatrix} \begin{bmatrix} 7.343 & 0 \\ 0 & 0.272 \end{bmatrix} \begin{bmatrix} -0.794 & -0.608 \\ 0.608 & -0.794 \end{bmatrix}^*$$

largest gain appears between the input direction $\begin{bmatrix} -0.794 \\ -0.608 \end{bmatrix}$ and the output direction $\begin{bmatrix} -0.872 \\ -0.490 \end{bmatrix}$ in the input, the largest value -0,794 is associated with input 1 and in the output, the largest value is -0.872, associated with output 1, so the recommended pairing is input 1 - output 1.

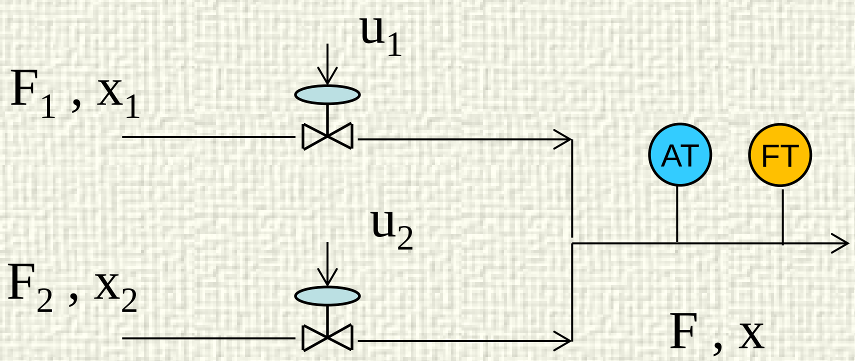
Then the procedure is repeated with the second largest gain (second column) being the largest values the ones corresponding to input 2 and output 2

Selecting variables with C_n

$$\text{condition number} = C_n = \frac{\overline{\sigma}(G)}{\underline{\sigma}(G)}$$

Multivariable systems with large condition number presents pairs with very large or very small gains that will make difficult the design of a good control system.
So, may be that only a subset of the inputs and outputs should be selected for control. This subset can be selected using the C_n

Example: Mixing streams



For $F_1 = 3, \quad F_2 = 2$

$x_1 = 0.7 \quad x_2 = 0.2$

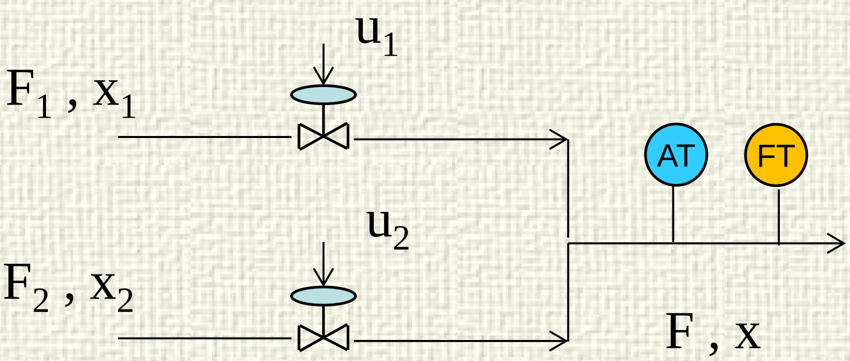
The steady state is : F 5, x = 0.5

$$G(0) = \begin{bmatrix} \left.\dfrac{F_2(x_1 - x_2)}{(F_1 + F_2)^2}\right|_{ss} & -\left.\dfrac{F_1(x_1 - x_2)}{(F_1 + F_2)^2}\right|_{ss} \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/25 & -1.5/25 \\ 1 & 1 \end{bmatrix}$$

$$\mathrm{svd}\left(\begin{bmatrix} 0.04 & -0.06 \\ 1 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0.01 & -0.9999 \\ -0.9999 & -0.01 \end{bmatrix} \begin{bmatrix} 1.4143 & 0 \\ 0 & 0.0707 \end{bmatrix} \begin{bmatrix} -0.7068 & -0.7075 \\ -0.7075 & 0.7068 \end{bmatrix}$$

$C_n = 1.4143/0.0707 = 20.0$

Example: mixing streams



$$U = \begin{bmatrix} 0.01 & -0.9999 \\ -0.9999 & -0.01 \end{bmatrix}$$

$$V = \begin{bmatrix} -0.7068 & -0.7075 \\ -0.7075 & 0.7068 \end{bmatrix}$$

Largest singular value 1.4143

$$U = \begin{bmatrix} 0.01 & -0.9999 \\ -0.9999 & -0.01 \end{bmatrix}$$

$$V = \begin{bmatrix} -0.7068 & -0.7075 \\ -0.7075 & 0.7068 \end{bmatrix}$$

→ x controlled with F_2

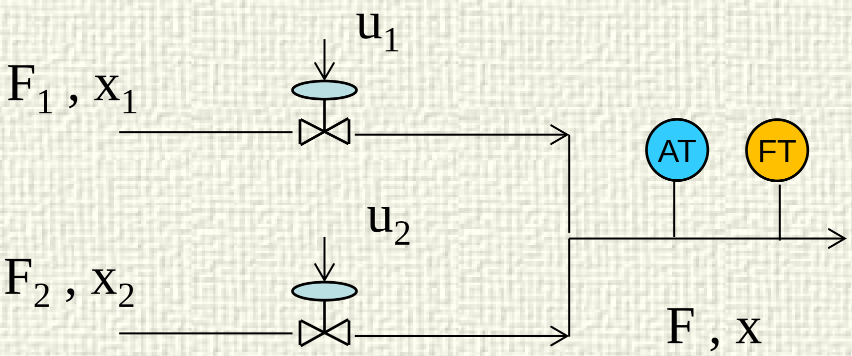
$$U = \begin{bmatrix} 0.01 & -0.9999 \\ -0.9999 & -0.01 \end{bmatrix}$$

$$V = \begin{bmatrix} -0.7068 & -0.7075 \\ -0.7075 & 0.7068 \end{bmatrix}$$

Second largest singular value 0.0707

F controlled with F_1 ←

Example: mixing streams



$$G(0) = \begin{bmatrix} 0.04 & -0.06 \\ 1 & 1 \end{bmatrix}$$

$$\text{Niederlinski index} = \frac{\det(G(0))}{0.04 * 1} = 0.25$$

$$G(0) = \begin{bmatrix} 0.04 & -0.06 \\ 1 & 1 \end{bmatrix}$$

$$RGA = \begin{bmatrix} 0.6 & 0.4 \\ 0.4 & 0.6 \end{bmatrix}$$

RGA analysis recommends the same pairing: (but not always)

F controlled with F_1

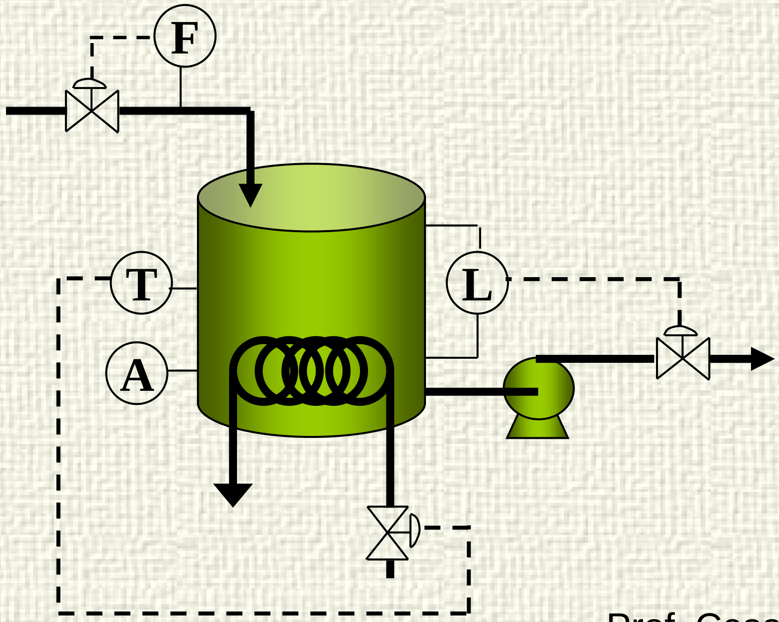
x controlled with F_2

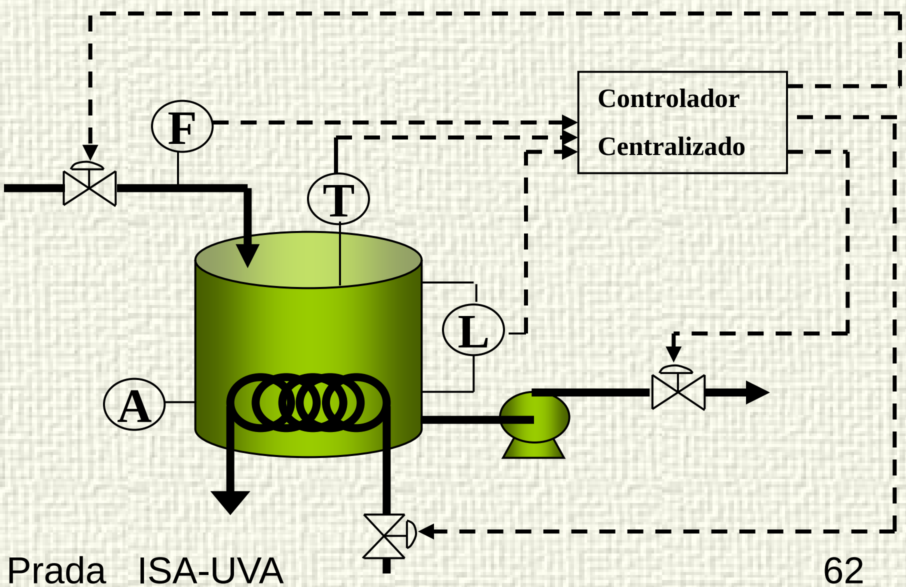
Designing control systems in multivariable processes

MULTILOOP vs Centralized

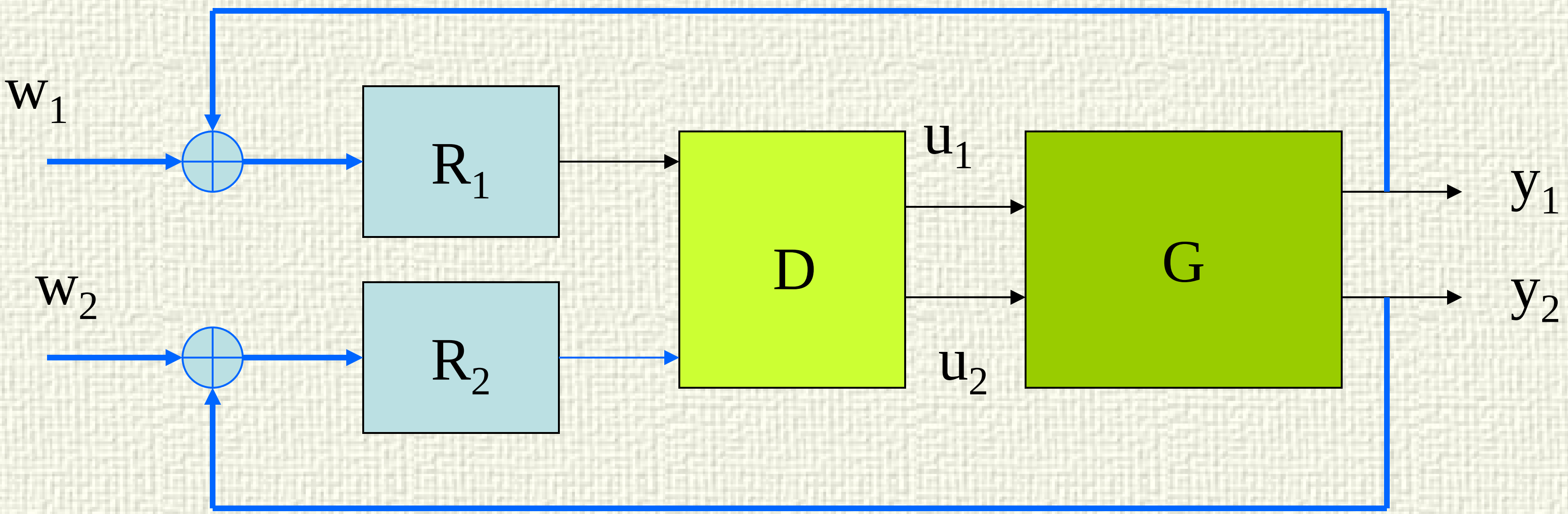
Multiloop: several independent PID controllers

One single multivariable controller



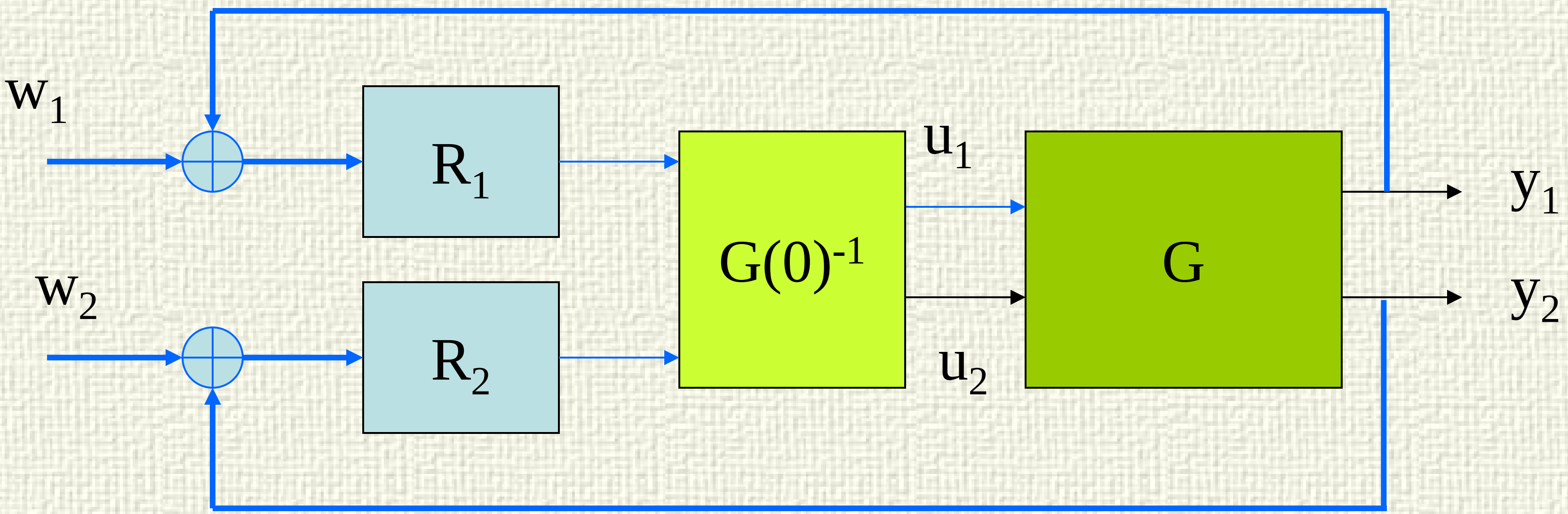


Decoupling



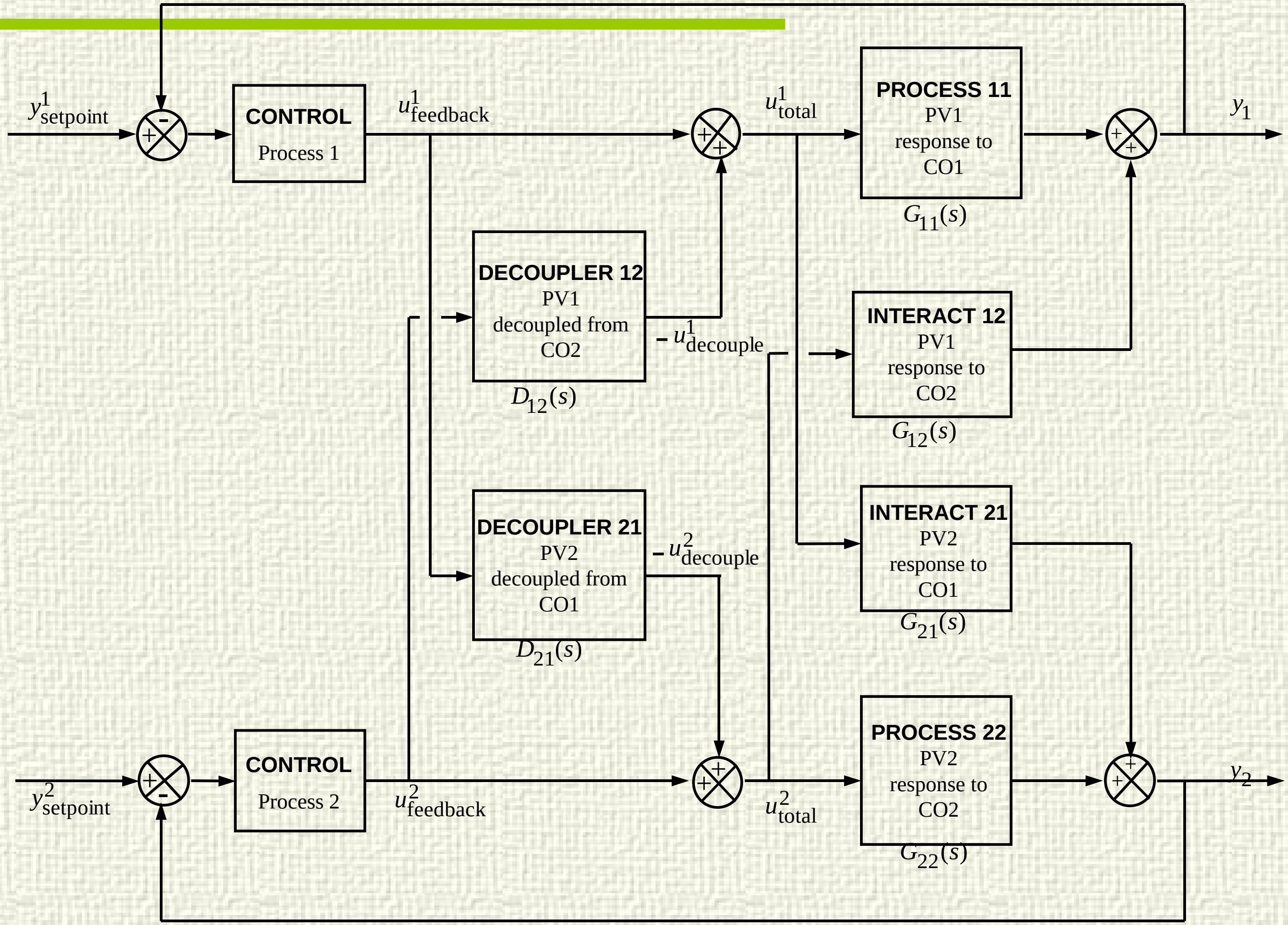
Find a matrix D such that GD behaves as a diagonal (or quasi diagonal) matrix, so that the interaction is cancelled

Steady state decoupling



If D is chosen as $\alpha G(0)^{-1}$,then $G(s)\ G(0)^{-1}$ is diagonal in steady state, so that there is no interaction at equilibrium. This decoupler is very easy to compute and implement because $G(0)^{-1}$ = inverse of the steady state gain matrix

Control structure with Decoupling

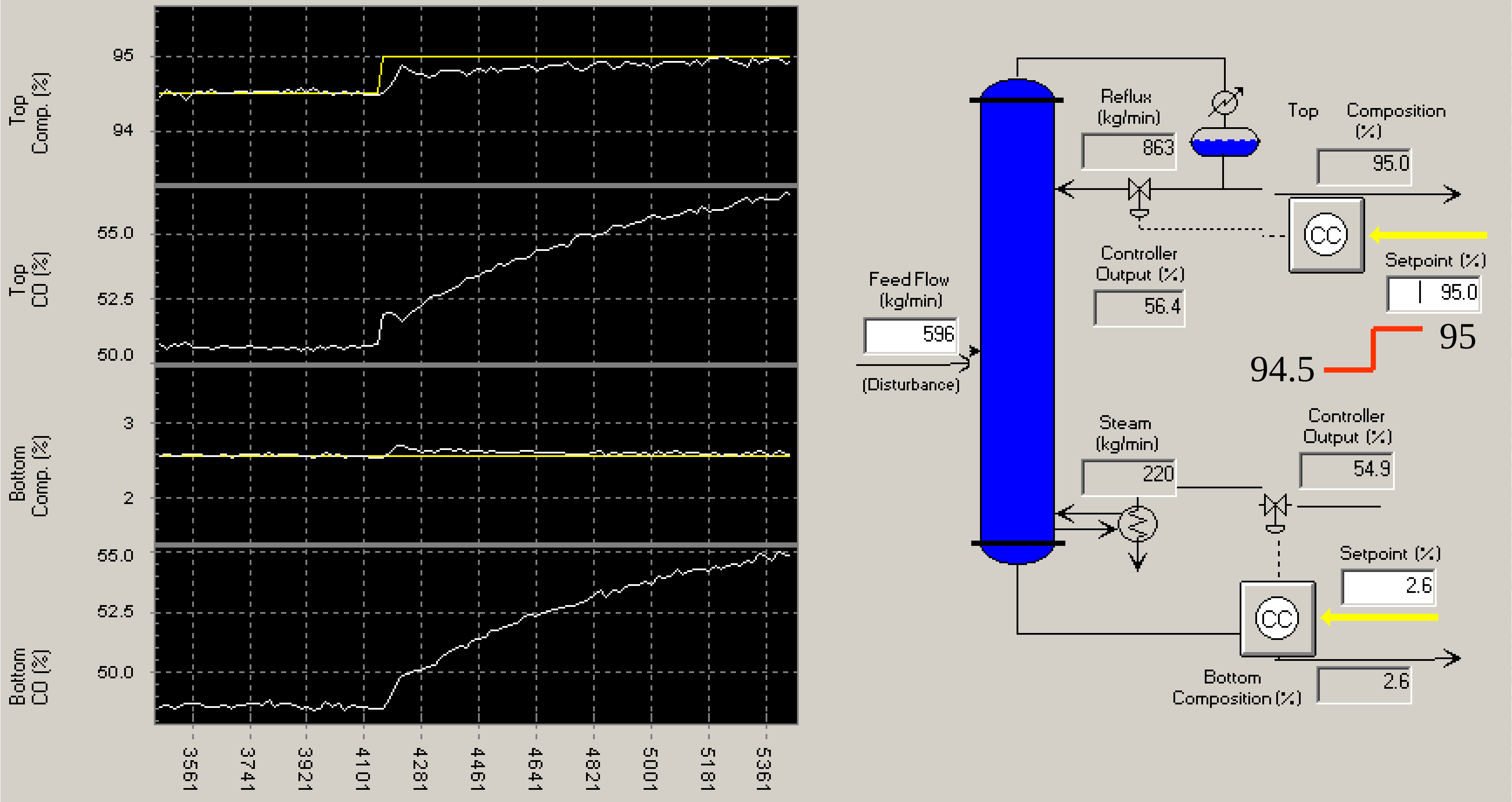


2x2 Multivariable Decoupling

- It requires 4 dynamical models:
  - Process 11 (how CO_1 influences PV_1)
  - Interact 12 (how CO_2 influences PV_1)
  - Interact 21 (how CO_1 influences PV_2)
  - Process 22 (how CO_2 influences PV_2)

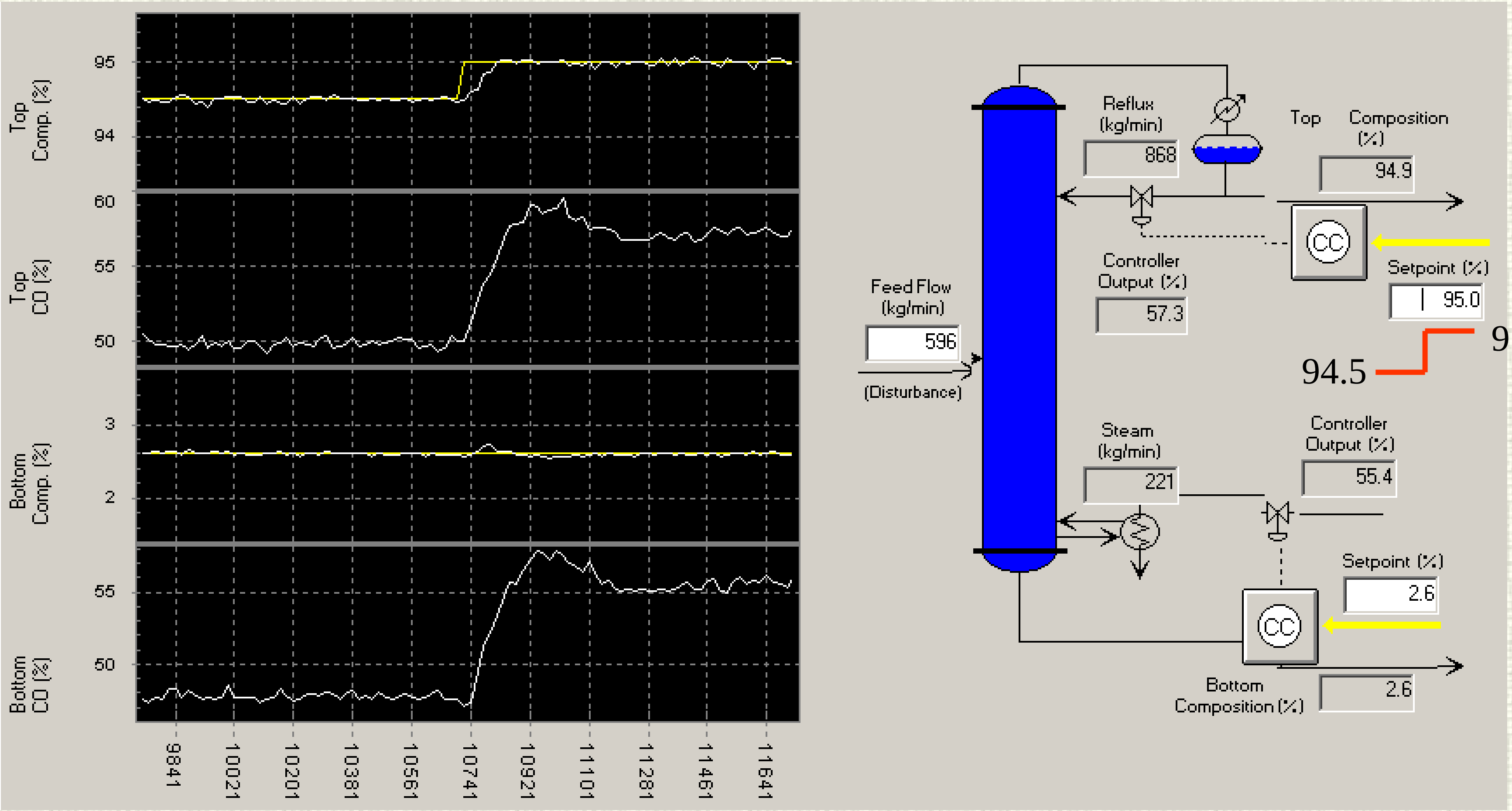
- These models must be obtained from validated experimental data.
- Loop decoupling is not use very often because it requires a certain effort in terms of modelling, tuning and maintenance, and MPC can provide better performance spending similar resources.

Interaction, Single loop PI



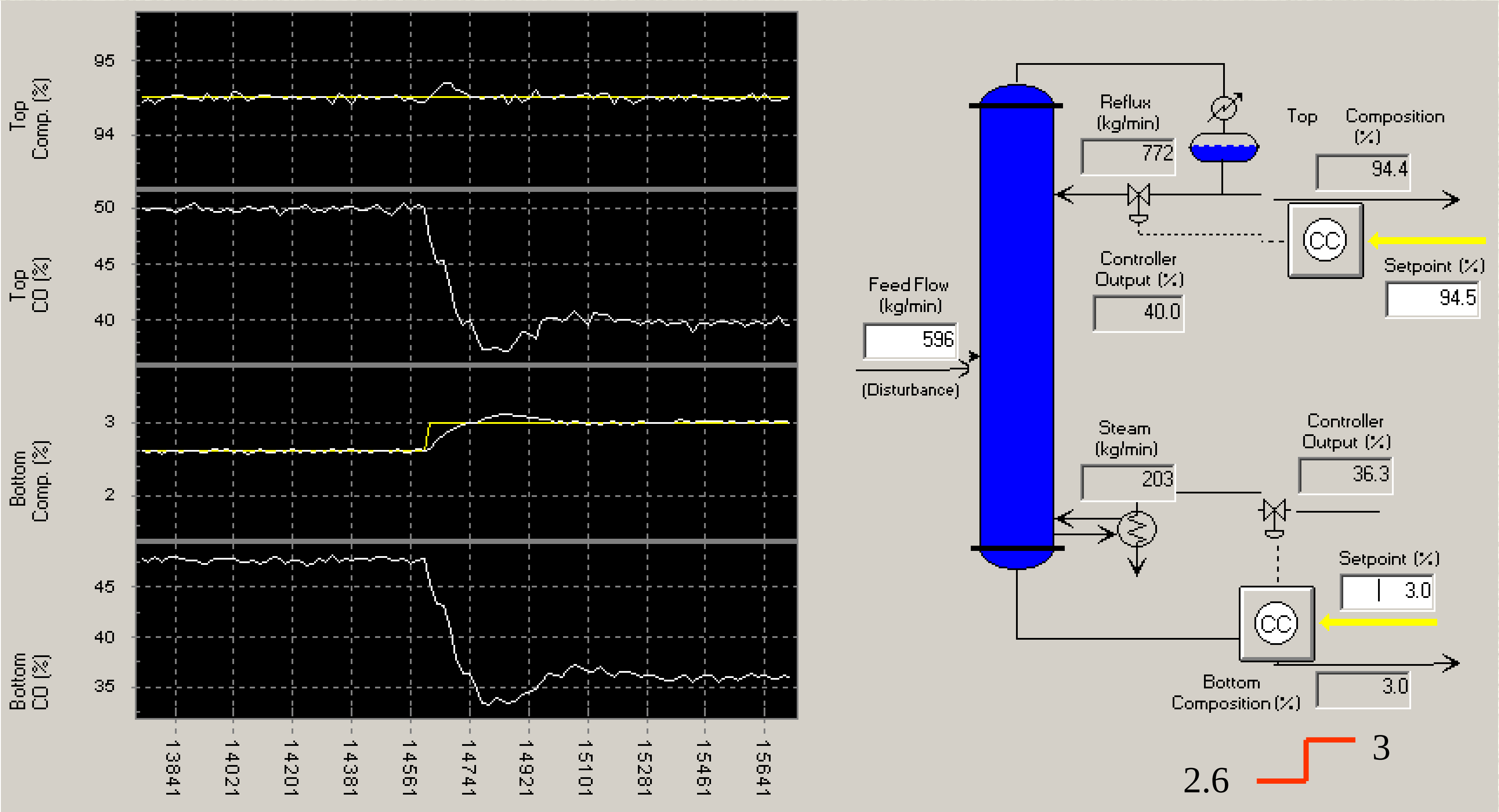
Controllers tuned independently with the other loop in manual

With decoupling

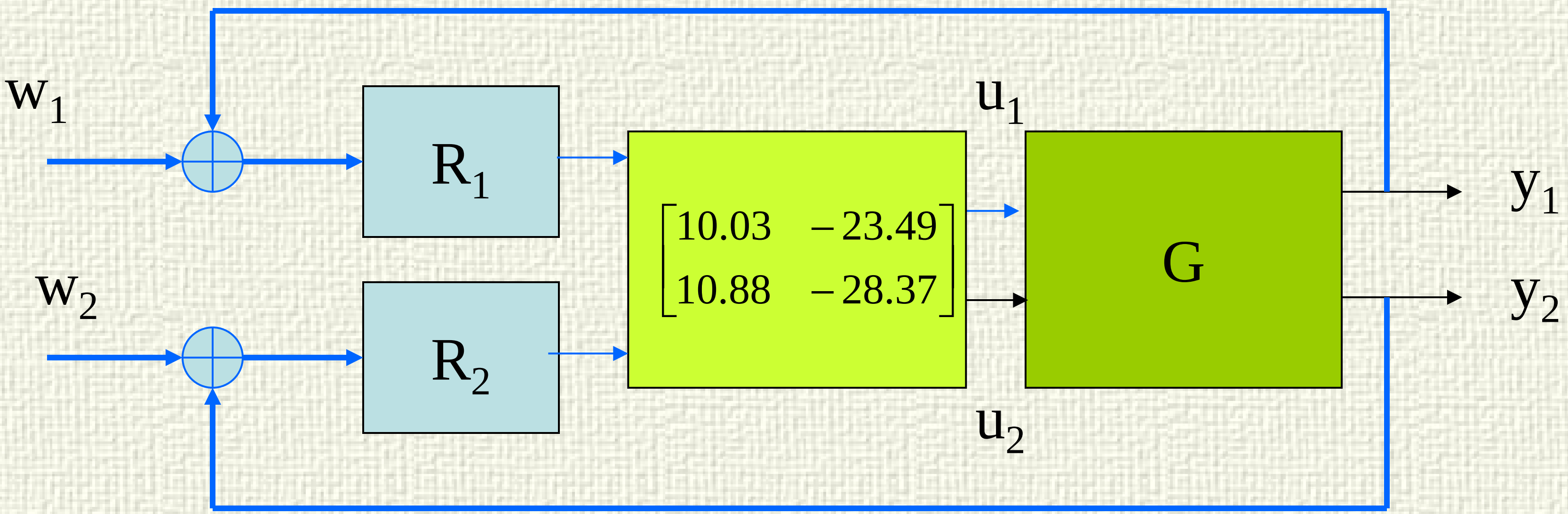


Same tuning

With decoupling

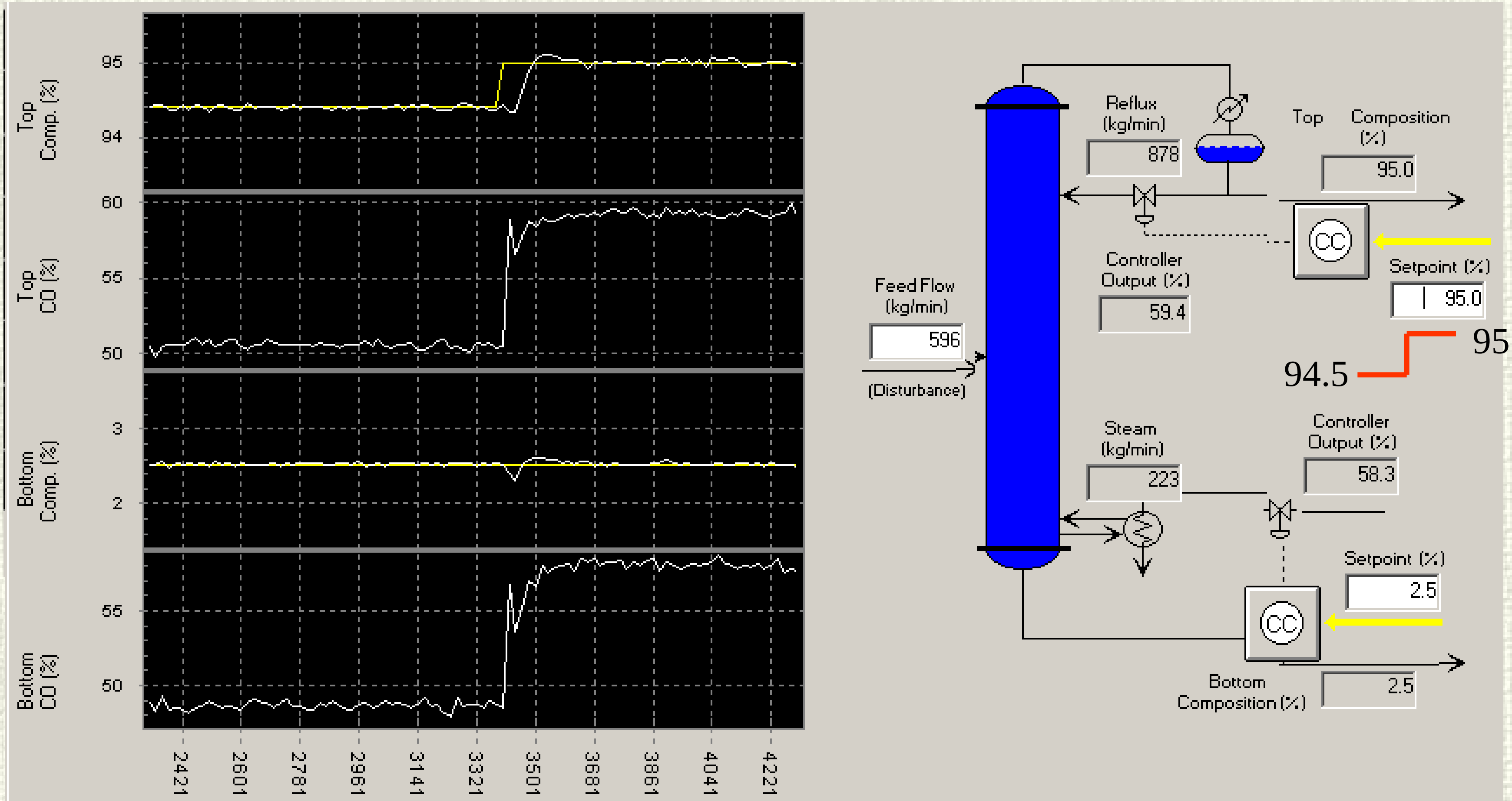


SS Decoupling



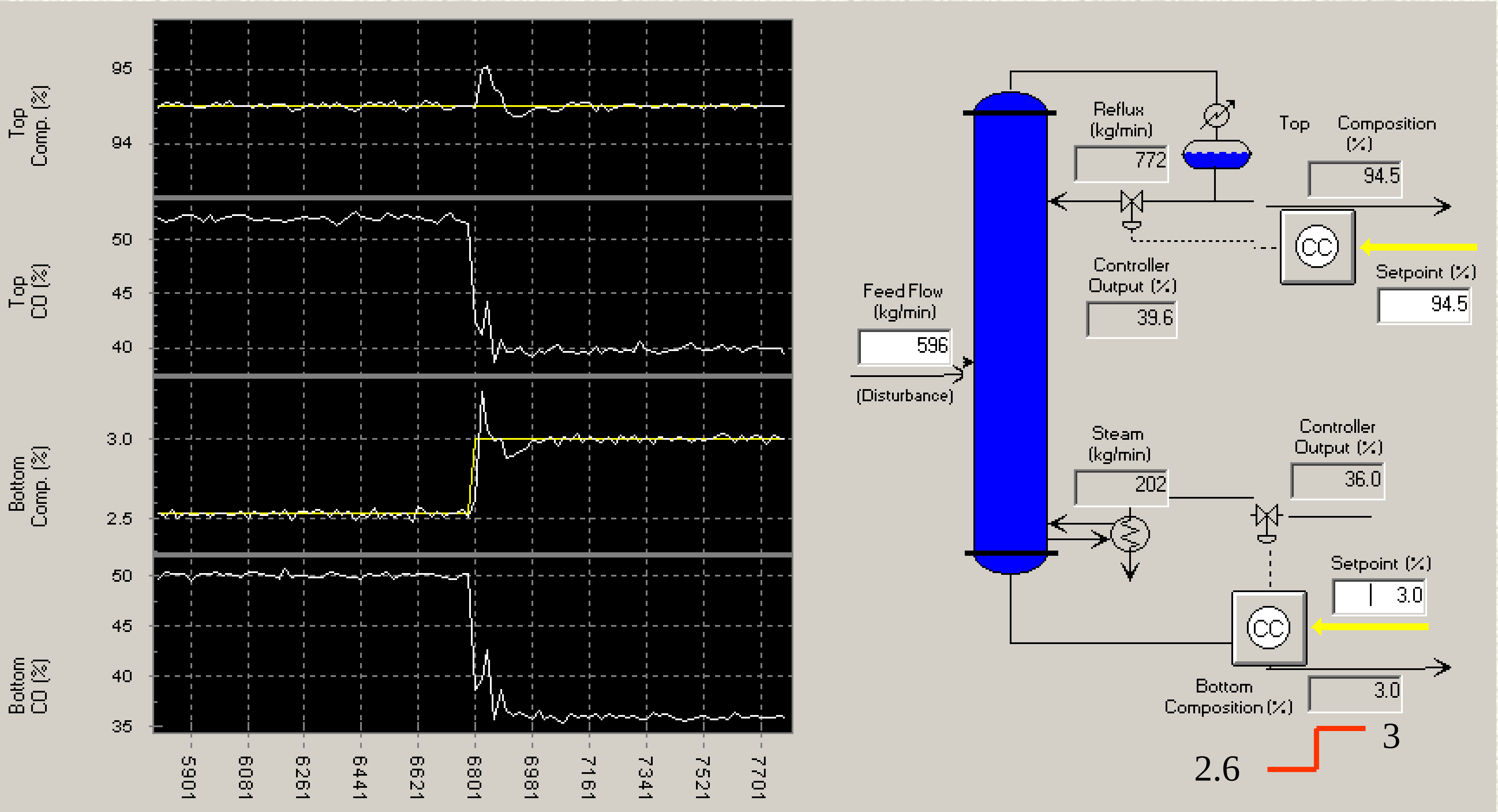
$$G(0)^{-1} = \begin{bmatrix} 0.99 & -0.82 \\ 0.38 & -0.35 \end{bmatrix}^{-1} = \begin{bmatrix} 10.03 & -23.49 \\ 10.88 & -28.37 \end{bmatrix}$$

With SS decoupling

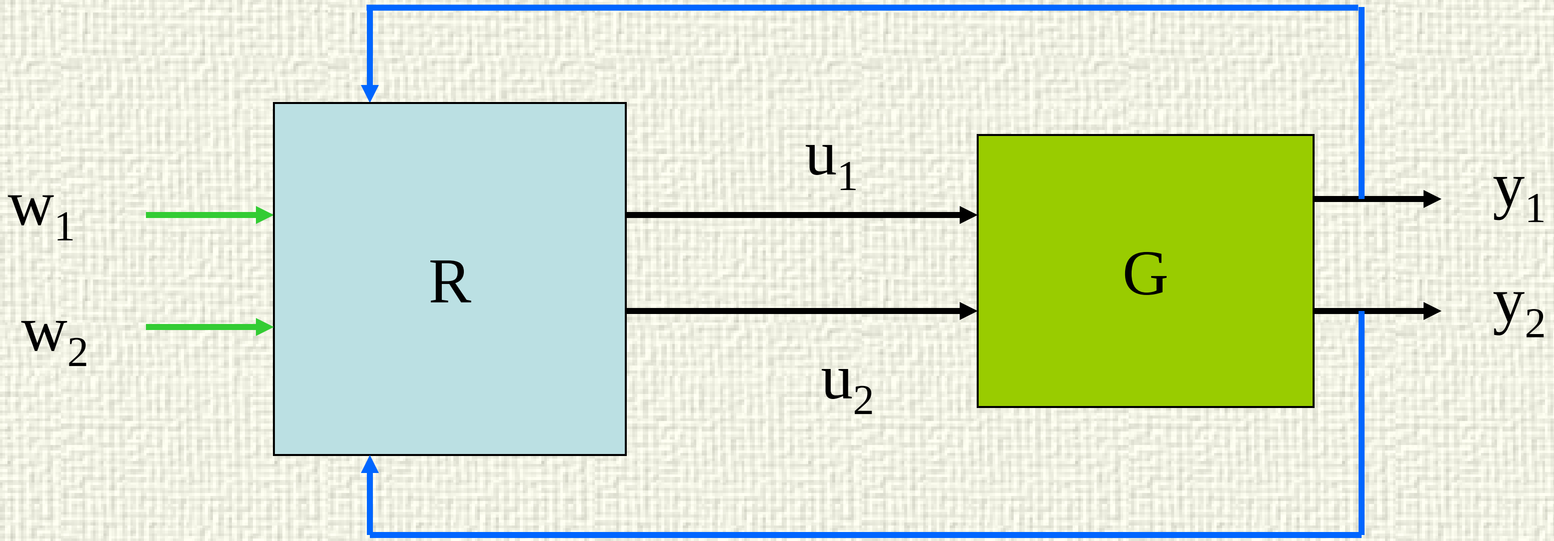


Same tuning

With SS decoupling

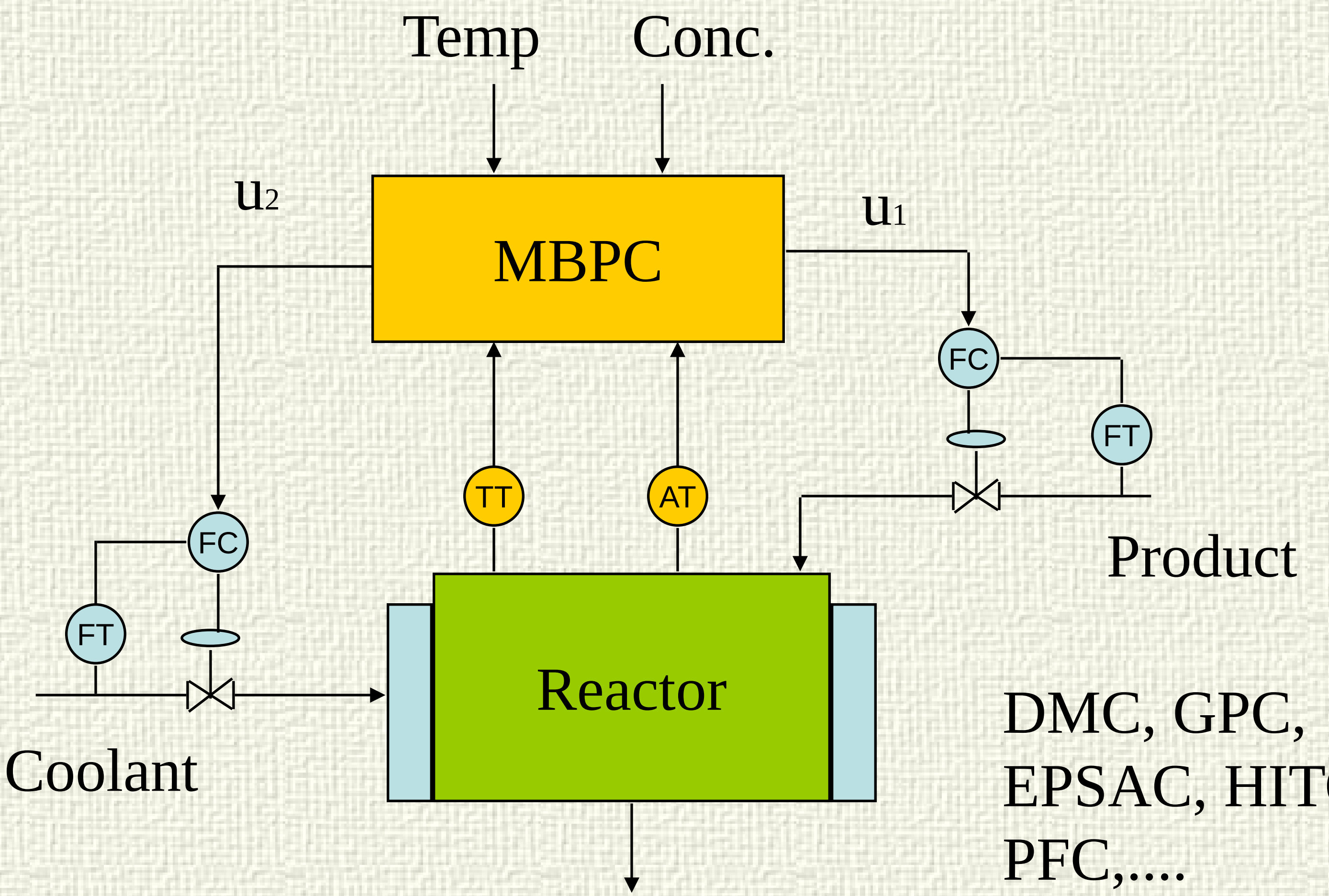


Multivariable Control



The controller receives signals from all controlled variables (and perhaps measurable disturbances) and computes simultaneously control actions for all actuators taking into account the interactions.

Multivariable Predictive Control MPC



DMC, GPC, EPSAC, HITO, PFC,....