Systems analysis in the frequency domain

Prof. Cesar de Prada
Dpt. Systems Engineering and Automatic Control
University of Valladolid, Spain
prada@autom.uva.es
http://www.isa.cie.uva.es/~prada/

The frequency domain

- The study of systems dynamics in the frequency domain allows to analyse and see the control systems from a different perspective
- Many aspects can be seen easier from this domain

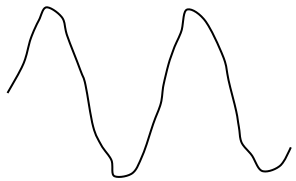
Aims / concepts

- ✓ Any signal can be described by its values over time or as a sum of sinusoidal signals of different amplitudes and frequencies.
- ✓ How is the response of a system when its input changes at different speeds (frequencies)?
- ✓ Signal filtering
- ✓ Analyse the dynamic behaviour from the point of view of the frequency domain

Outline

- ✓ Fourier transform
- ✓ Frequency response
- ✓ Signal filtering
- ✓ Closed loop stability in the frequency domain
- ✓ Delays
- ✓ Robustness

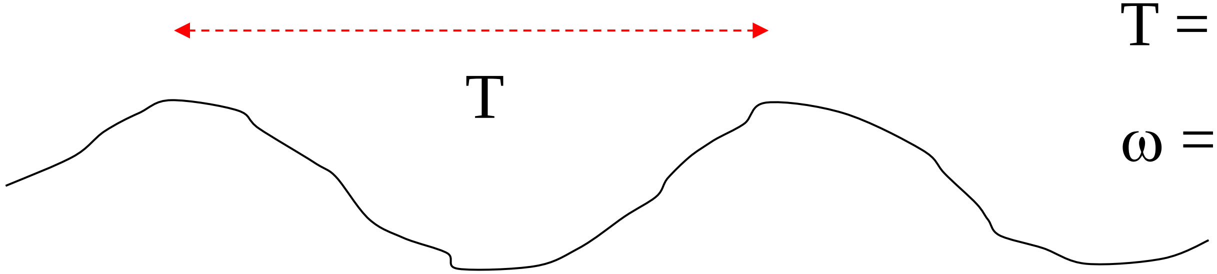
Sinusoidal signals



High frequency: high speed of change

T = period

ω = frequency

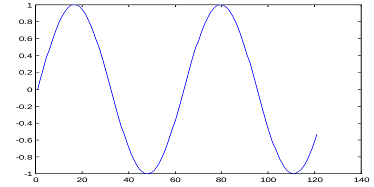


$\omega = 2\pi/T$ rad/time

$f = 1/T$ 1/time Hz

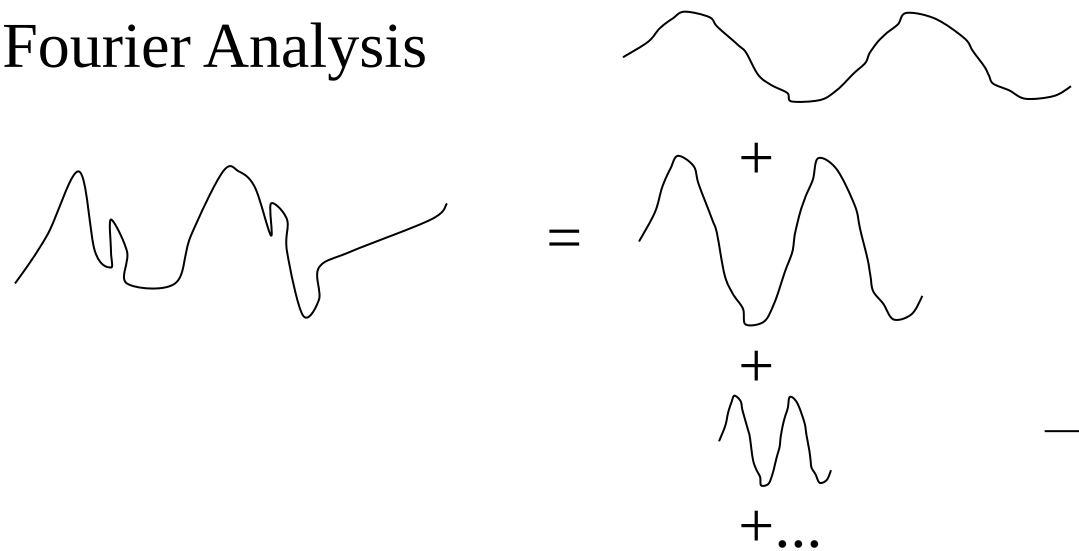
$u = A \sin(\omega t)$

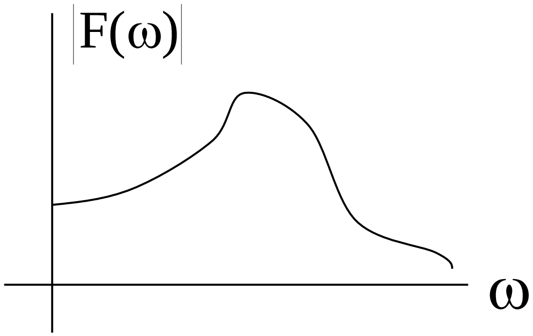
Low frequency: slow signal



Frequency components

Fourier Analysis



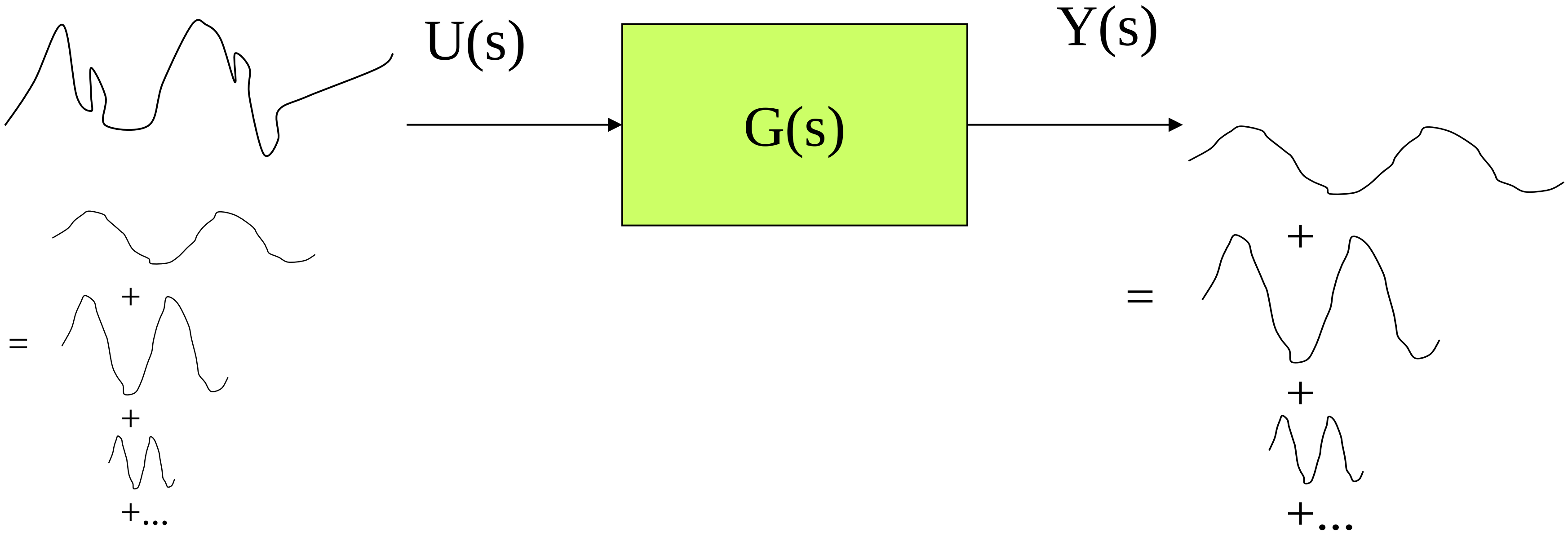


$$f(t) = \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega \qquad e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

Spectrum of f(t)

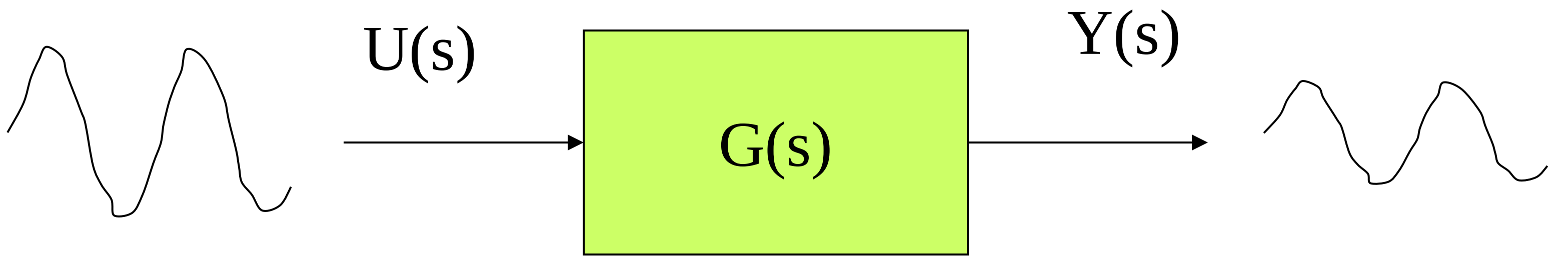
Any signal can be decomposed in the sum of infinite sinusoidal signals of different amplitudes and frequencies

Response of a system to an arbitrary input signal



The response of a system with TF G(s) to any input signal is the sum of the responses of the system to each of the sinusoidal signals that make up the input signal

Sinusoidal inputs



To study the response of a (stable) linear system G(s) against sinusoidal input signals at steady state.

Different frequencies = different speeds of change

$$Y(s) = G(s)\,U(s) \qquad U(s) = \frac{\omega A}{s^2 + \omega^2} \qquad G(s) = \frac{N(s)}{D(s)}$$

$$\lim_{s \to 0} sY(s)$$

Frequency response

$$Y(s) = G(s)U(s) = \frac{N(s)}{D(s)}\frac{\omega A}{s^2+\omega^2} = \frac{a}{s+j\omega} + \frac{\bar{a}}{s-j\omega} + \frac{b(s)}{D(s)}$$

$$\frac{N(s)}{D(s)}\frac{\omega A}{s^2+\omega^2} = \frac{a(s-j\omega)D(s) + \bar{a}(s+j\omega)D(s) + b(s)(s+j\omega)(s-j\omega)}{(s+j\omega)(s-j\omega)D(s)}$$

$$N(s)\omega A = a(s-j\omega)D(s) + \bar{a}(s+j\omega)D(s) + b(s)(s+j\omega)(s-j\omega)$$

$$\text{for } \; s = j\omega \qquad N(j\omega)\omega A = \bar{a}\,2j\omega D(j\omega) \qquad \bar{a} = \frac{AG(j\omega)}{2j}$$

$$\text{for } \; s = -j\omega \qquad N(-j\omega)\omega A = -a\,2j\omega D(-j\omega) \qquad a = \frac{-AG(-j\omega)}{2j}$$

Frequency response

$$y(t) = L^{-1}[Y(s)] = L^{-1}\left[\frac{a}{s + j\omega}\right] + L^{-1}\left[\frac{\bar{a}}{s - j\omega}\right] + L^{-1}\left[\frac{b(s)}{D(s)}\right]$$

$$y(t) = \frac{-AG(-j\omega)}{2j} e^{-j\omega t} + \frac{AG(j\omega)}{2j} e^{j\omega t} + \ldots\ldots$$

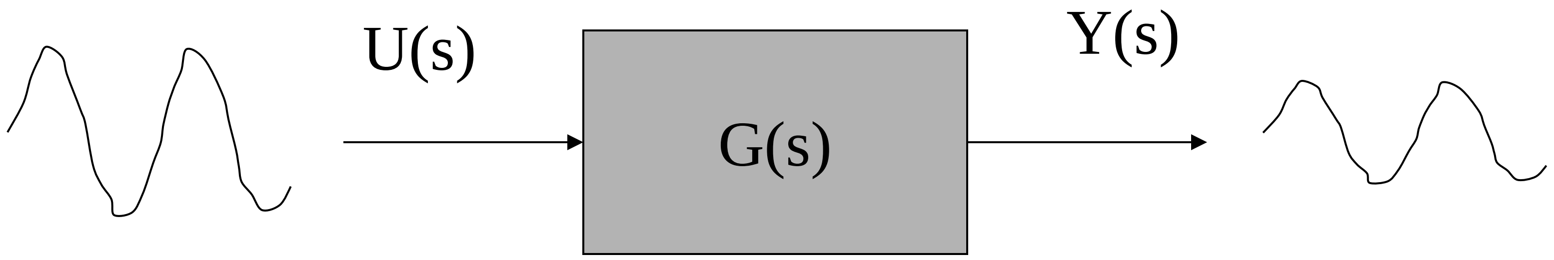
if D(s) is stable, in steady state:

$$y_\infty = \lim_{t\to\infty} y(t) = \frac{-AG(-j\omega)}{2j} e^{-j\omega t} + \frac{AG(j\omega)}{2j} e^{j\omega t}$$

$$y_\infty = \frac{-A|G(j\omega)|e^{-j\phi}}{2j} e^{-j\omega t} + \frac{A|G(j\omega)|e^{j\phi}}{2j} e^{j\omega t}$$

$$y_\infty = A|G(j\omega)| \frac{e^{j(\omega t+\phi)} - e^{-j(\omega t+\phi)}}{2j} = A|G(j\omega)| \mathrm{sen}(\omega t + \phi) \qquad \phi = \arg(G(j\omega))$$

Frequency response



$$u(t) = A\sin(\omega t)$$

$$y_\infty = A|G(j\omega)|\sin(\omega t + \phi) \qquad \phi = \arg(G(j\omega))$$

The response is also a sinusoidal signal of the same frequency ω, but amplified (or attenuated) by a factor $|G(j\omega)|$ and phase-shifted by an angle $\phi = \arg(G(j\omega))$, both of which depend on ω

CStation

Frequency response

The values of the gain factor $|G(j\omega)|$ and the phase shift $\phi = \arg(G(j\omega))$ depend only on $G(j\omega)$ and can be represented as a function of the frequency ω using several types of diagrams. For this purpose, s must be substituted by $j\omega$ in $G(s)$, and the module and argument of the complex number $G(j\omega)$ must be computed for different values of the frequency ω

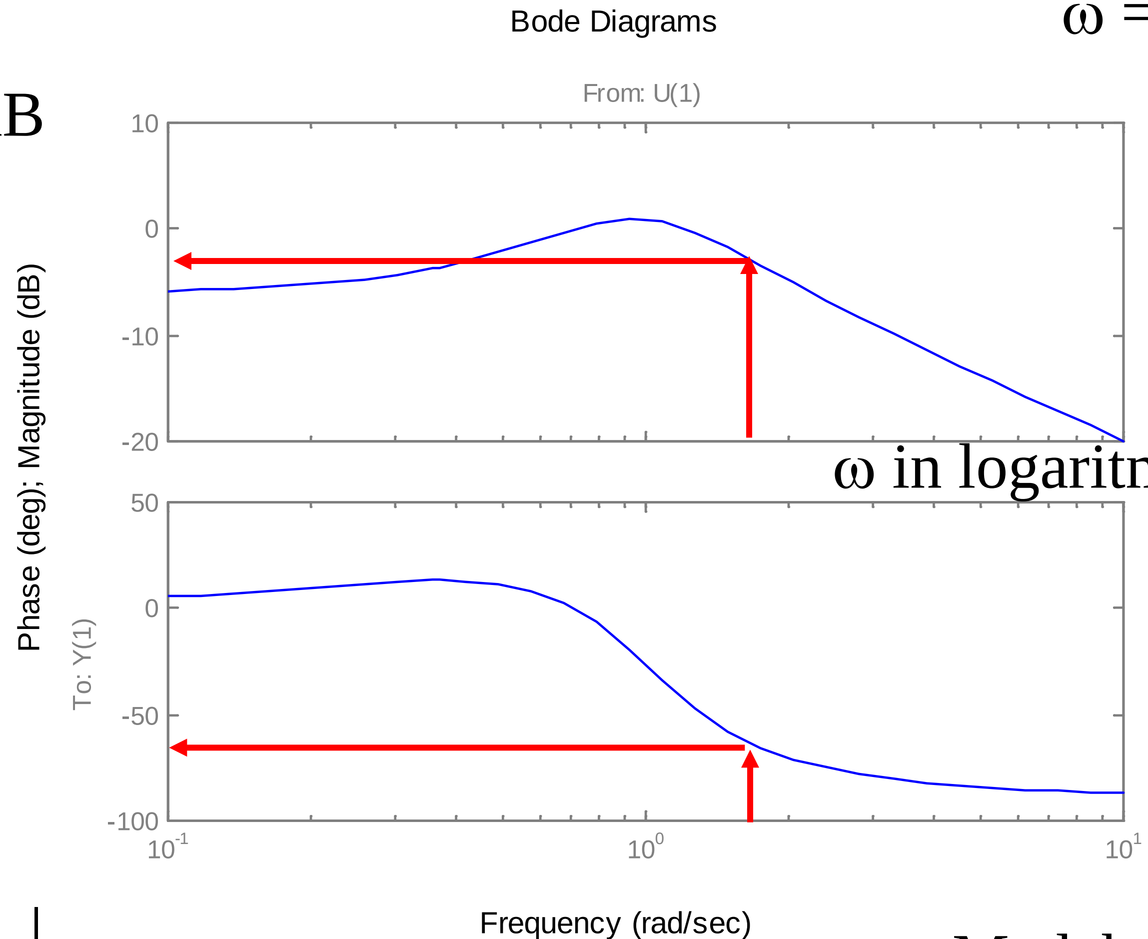
$$G(s) = \frac{(2s+1)}{s^2+3s+2} \Rightarrow G(j\omega) = \frac{(2j\omega+1)}{j^2\omega^2+3j\omega+2} = \frac{(2j\omega+1)}{2-\omega^2+3j\omega}$$

$$|G(j\omega)| = \frac{\sqrt{1+4\omega^2}}{\sqrt{(2-\omega^2)^2+9\omega^2}} \qquad \arg(G(j\omega)) = \operatorname{arctg} 2\omega - \operatorname{arctg}\frac{3\omega}{2-\omega^2}$$

Bode Diagram

$\omega = (0,\infty)$

$|G(j\omega)|$ in dB



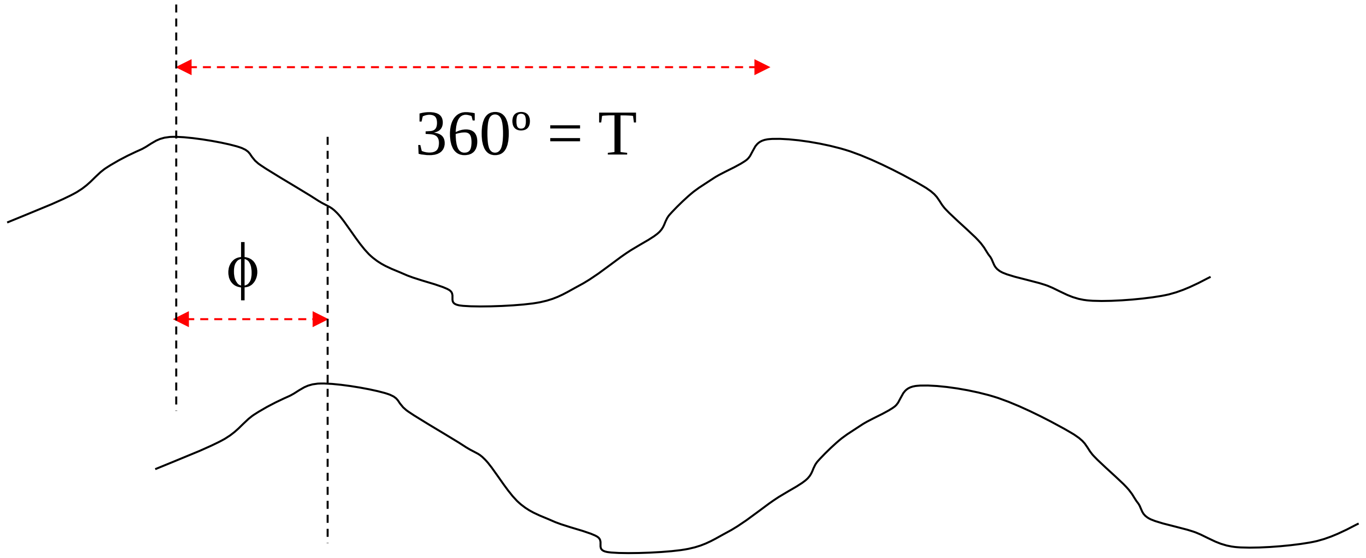
ω in logaritmic scale

$\arg(G(j\omega))$ in degrees

dB = 20log | . |

Matlab: bode(sys)

Phase shift in degrees



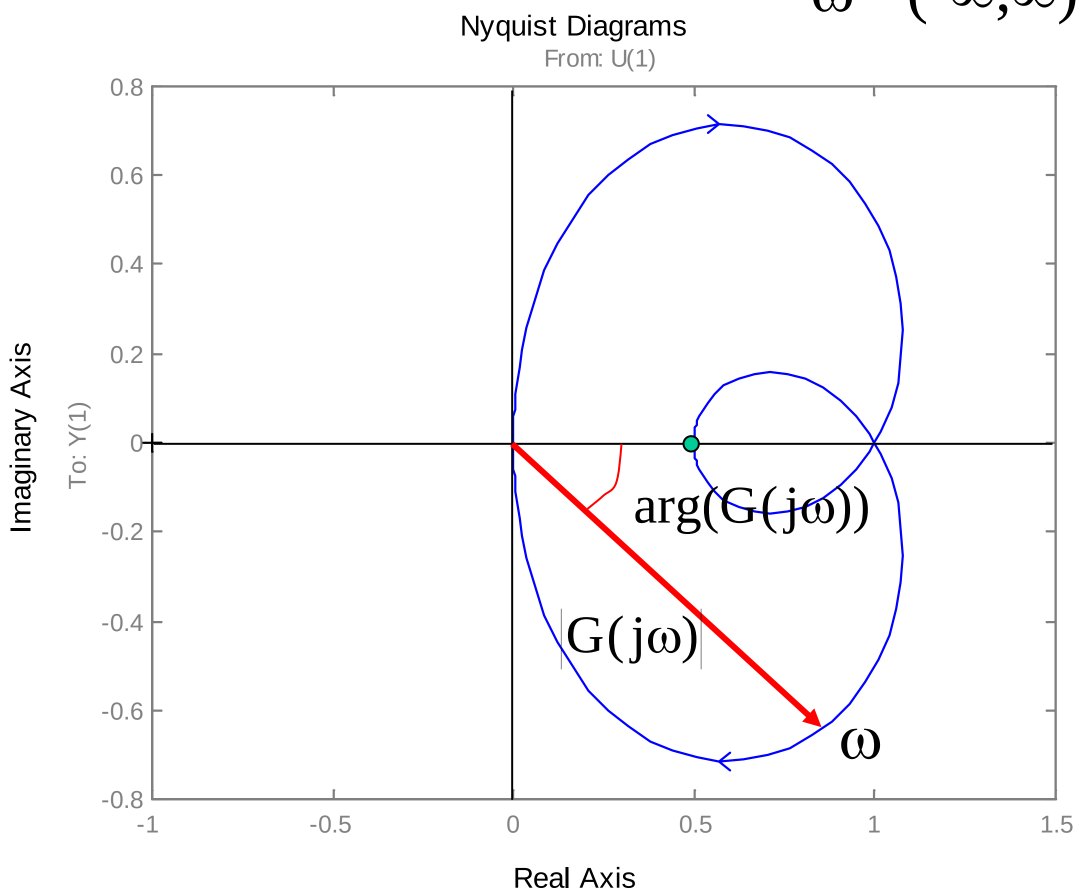
A phase shift φ in degrees can be translated into a delay time in time units as φ T/360

Nyquist Diagram

$\omega = (-\infty, \infty)$

For every value of ω, the corresponding module and argument of G(jω) is plotted

Polar diagram parameterized in frequency

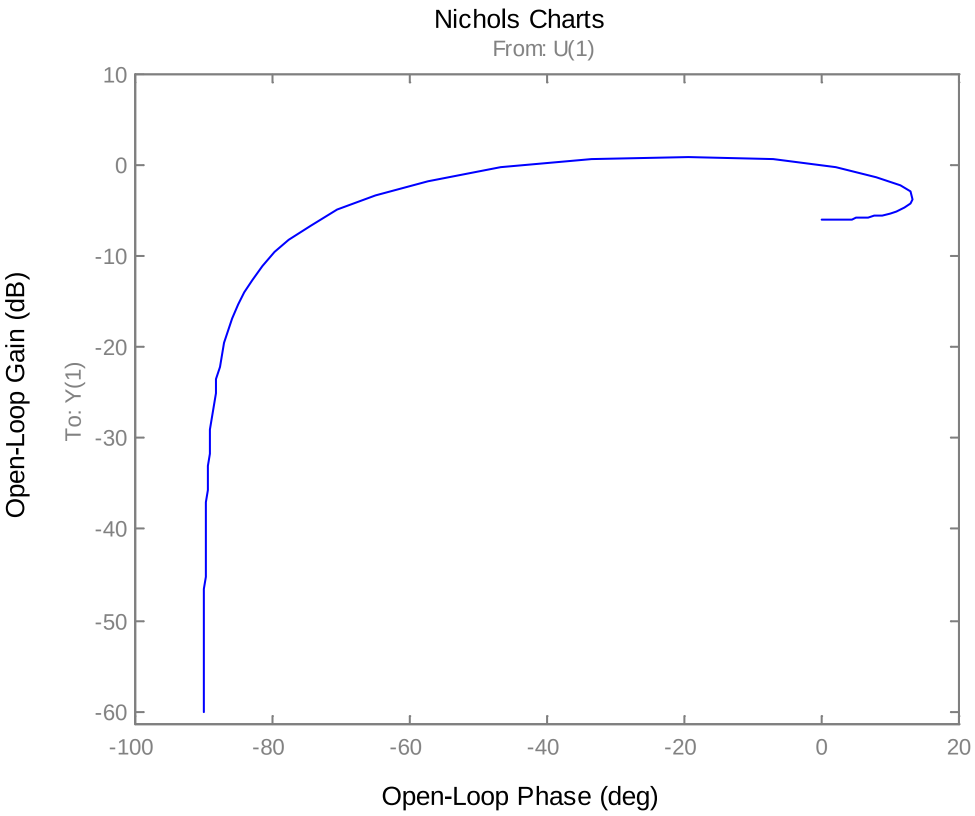


Matlab: nyquist(sys)

Nichols Diagram

The values of |G(jω)| in dB, are plotted as a function of arg(G(jω)) in degrees

Matlab: nichols(sys)



Why logaritmic diagrams?

$$G(s) = \frac{Ke^{-ds}(cs+1)(\ldots)}{s(\tau s+1)(\ldots)} \qquad G(j\omega) = \frac{Ke^{-dj\omega}(cj\omega+1)(\ldots)}{j\omega(\tau j\omega+1)(\ldots)}$$

$$20\log|G(j\omega)| = 20\log\left|\frac{Ke^{-dj\omega}(cj\omega+1)(\ldots)}{j\omega(\tau j\omega+1)(\ldots)}\right| =$$

$$= 20\log|K| + 20\log\left|e^{-dj\omega}\right| + 20\log|cj\omega+1| + \ldots + 20\log\left|\frac{1}{j\omega}\right| + 20\log\left|\frac{1}{\tau j\omega+1}\right| + \ldots$$

In dB, the diagram of $|G(j\omega)|$ can be obtained by superposition of the diagrams of the elementary terms corresponding to every pole, zero, gain and delay

$$\arg(G(j\omega)) = \arg(K) + \arg(e^{-j\omega d}) + \arg(cj\omega+1) + \ldots + \arg(1/j\omega) + \arg(1/(\tau j\omega+1)) + \ldots$$

Bode: single pole

$$20\log\left|\frac{1}{j\omega\tau+1}\right| = -20\log\sqrt{1+\tau^2\omega^2} =$$

$$= -10\log(1+\tau^2\omega^2)$$

monotonously decreasing

for $\omega \to 0 \qquad -10\log(1+\tau^2\omega^2) \to 0$

for $\omega \to \infty$

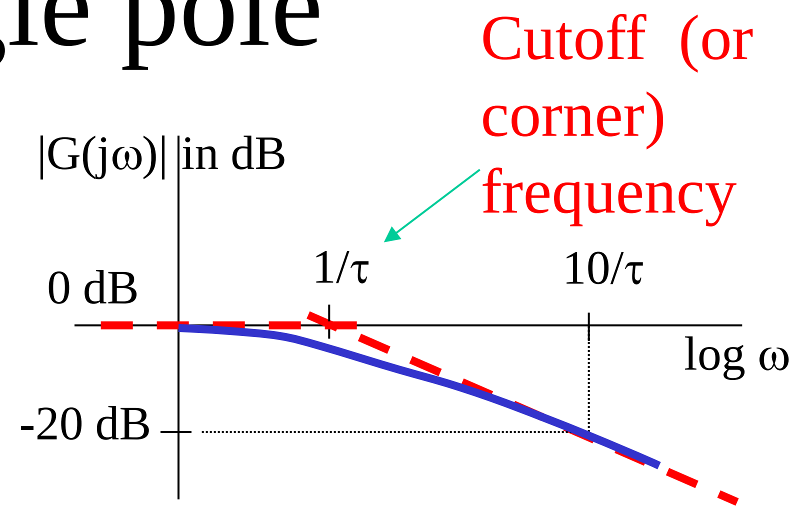
$$-10\log(1+\tau^2\omega^2) \to -20\log\tau - 20\log\omega$$

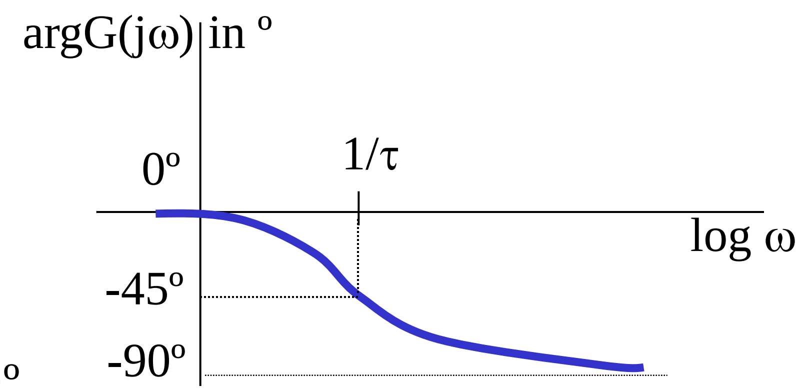
straight line of slope - 20dB and

passing through $(\omega = 1/\tau \,, 0\text{ dB})$

$$\arg\left(\frac{1}{j\omega\tau+1}\right) = -\text{arctg}(\omega\tau)\begin{cases} \omega \to 0 & \phi \to 0 \\ \omega \to \infty & \phi \to -90^{o} \end{cases}$$

monotonously decreasing, $\phi = 45^{o}$ at $\omega = 1/\tau$

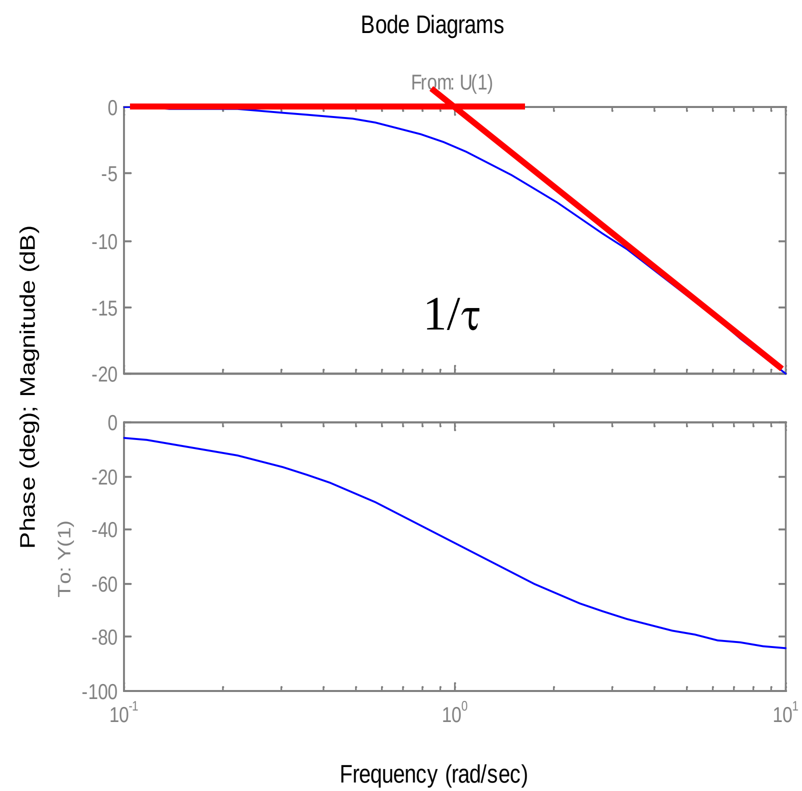




Bode: single pole

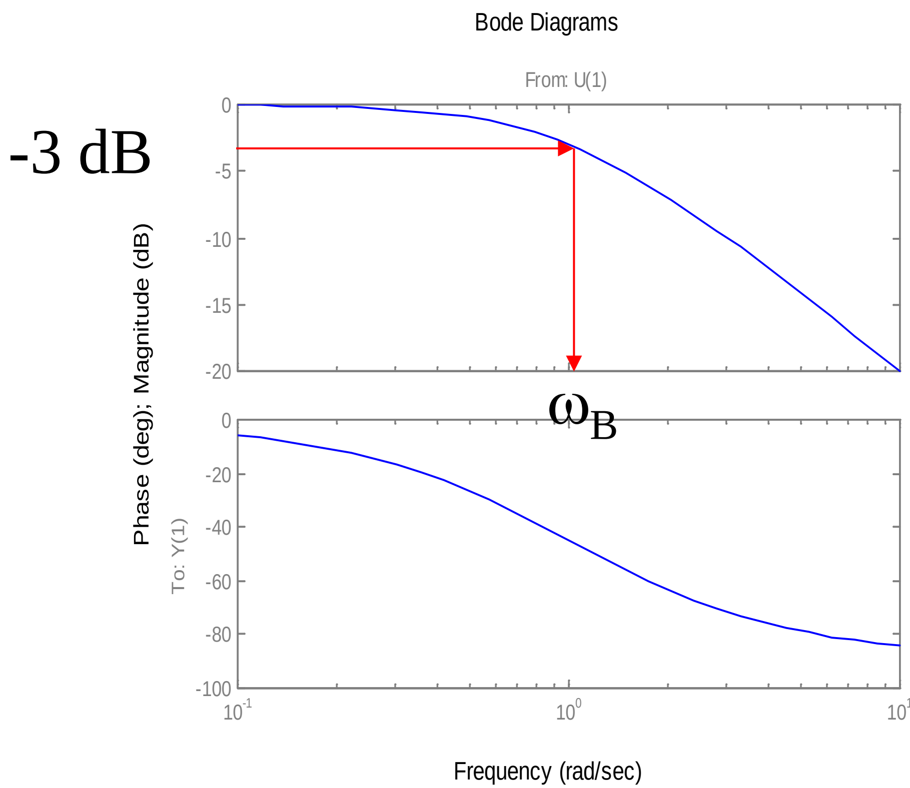
Small attenuation until the cutoff frequency 1/τ, but then progressively increasing at -20dB/decade

Slow systems (τ large) have small cutoff frequencies, so that fast input changes will be lessen at the system output. Fast systems, by the contrary, respond to a wider range of input signals



Bandwidth

At frequencies below ω_B the attenuation is less than -3 dB (or the output power is more then one half of the input power). It gives a measure of the range of speeds of change at the input to which the system responds without too much attenuation.



$$\log(\sqrt{1/2}) = 3$$

Nyquist Diagram

$$\left|\frac{1}{j\omega\tau+1}\right| = \frac{1}{\sqrt{1+\tau^2\omega^2}}$$

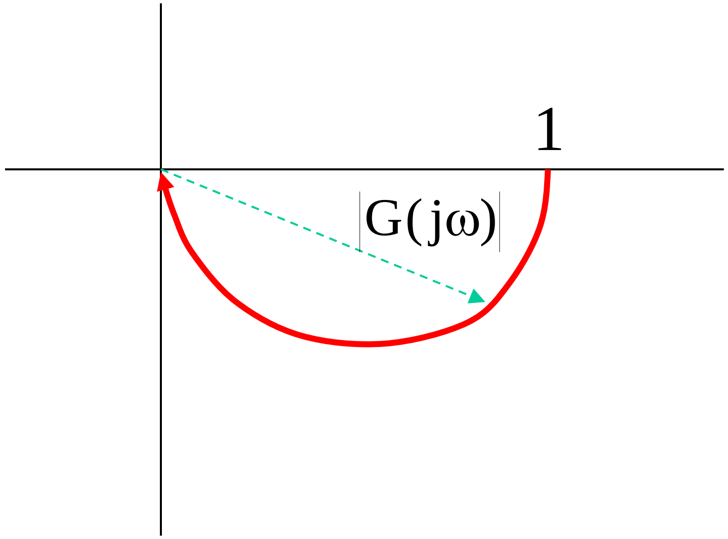
monotonously decreasing

$$\text{for } \omega \to 0 \qquad \left|\frac{1}{j\omega\tau+1}\right| \to 1$$

$$\text{for } \omega \to \infty \qquad \left|\frac{1}{j\omega\tau+1}\right| \to 0$$

$$\arg\left(\frac{1}{j\omega\tau+1}\right) = -\text{arctg}(\omega\tau)\begin{cases} \omega \to 0 & \phi \to 0 \\ \omega \to \infty & \phi \to -90^o \end{cases}$$

monotonously decreasing, $\phi = 45^o$ at $\omega = 1/\tau$



Another branch for -∞ to 0

Bode: Single zero

$$20\log\left|j\omega c+1\right| = 20\log\sqrt{1+c^2\omega^2} =$$

$$= 10\log(1+c^2\omega^2)$$

monotonously decreasing

for $\omega \to 0 \qquad 10\log(1+c^2\omega^2) \to 0$

for $\omega \to \infty$

$$10\log(1+c^2\omega^2) \to 20\log c + 20\log\omega$$

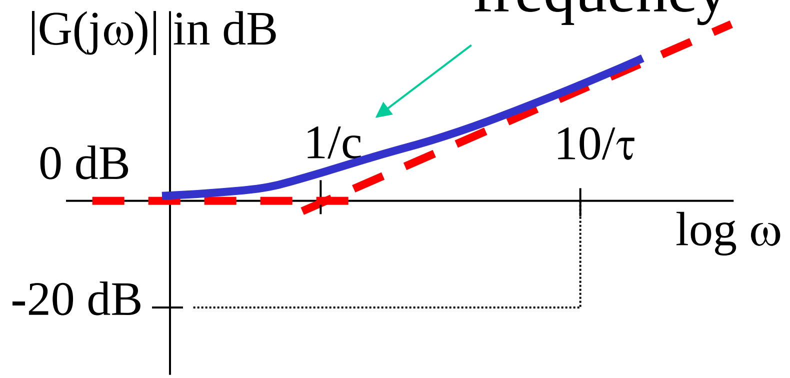
straight line of slope 20dB

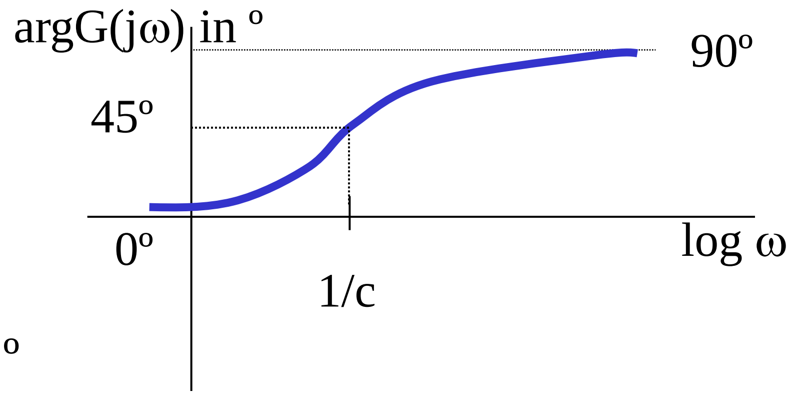
passing through $(\omega = 1/\tau\,,\,0\text{ dB})$

$$\arg\left(j\omega c+1\right) = \text{arctg}(\omega c)\begin{cases} \omega \to 0 & \phi \to 0 \\ \omega \to \infty & \phi \to 90^o \end{cases}$$

monotonousty increasing, $\phi = 45^o$ at $\omega = 1/c$

Cutoff frequency





High frequencies are amplified

Bode: double pole

$$20\log\left|\frac{1}{(j\omega\tau+1)^2}\right| = -20\log\left(1+\tau^2\omega^2\right)$$

monotonously decreasing

for $\omega \to 0 \qquad -20\log(1+\tau^2\omega^2) \to 0$

for $\omega \to \infty$

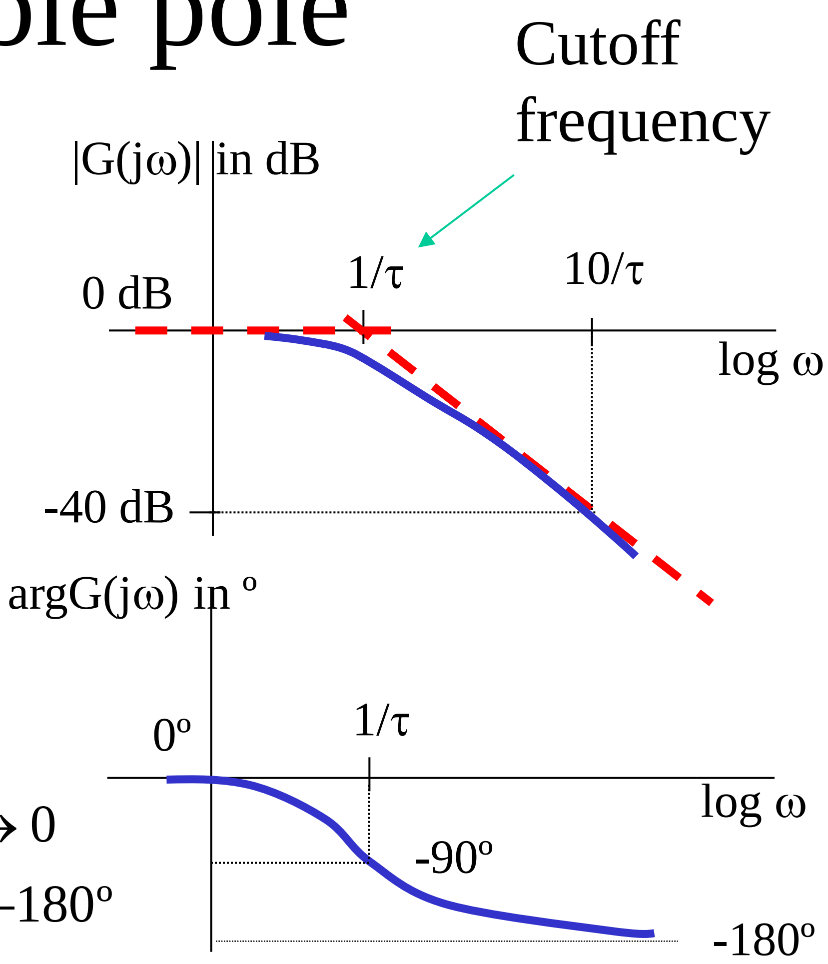
$-20\log(1+\tau^2\omega^2) \to -40\log\tau - 40\log\omega$

straight line of slope - 40dB

passing through $(\omega = 1/\tau\,, 0\text{ dB})$

$$\arg\left(\frac{1}{(j\omega\tau+1)^2}\right) = -2\text{arctg}(\omega\tau)\begin{cases} \omega \to 0 & \phi \to 0 \\ \omega \to \infty & \phi \to -180^o \end{cases}$$

monotonously decreasing, $\phi = -90^o$ at $\omega = 1/\tau$



Nyquist Diagram

$$\left|\frac{1}{(j\omega\tau+1)^2}\right| = \frac{1}{1+\tau^2\omega^2}$$

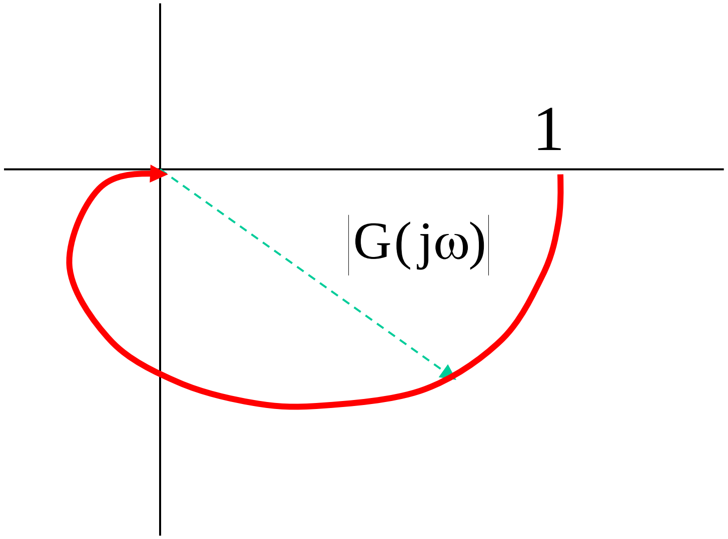
monotonously decreasing

for $\omega \to 0 \qquad \left|\frac{1}{(j\omega\tau+1)^2}\right| \to 1$

for $\omega \to \infty \qquad \left|\frac{1}{(j\omega\tau+1)^2}\right| \to 0$

$$\arg\left(\frac{1}{(j\omega\tau+1)^2}\right) = -2\,\mathrm{arctg}(\omega\tau)\begin{cases}\omega \to 0 & \phi \to 0 \\ \omega \to \infty & \phi \to -180^o\end{cases}$$

monotonously decreasing, $\phi = 90^o$ at $\omega = 1/\tau$



Bode: complex conjugate poles

$$\frac{\omega_n^2}{s^2+2\delta\omega_n s+\omega_n^2} \rightarrow \frac{1}{\left(\frac{j\omega}{\omega_n}\right)^2+2\delta\frac{j\omega}{\omega_n}+1} = \frac{1}{1-\frac{\omega^2}{\omega_n^2}+j\frac{2\delta\omega}{\omega_n}}$$

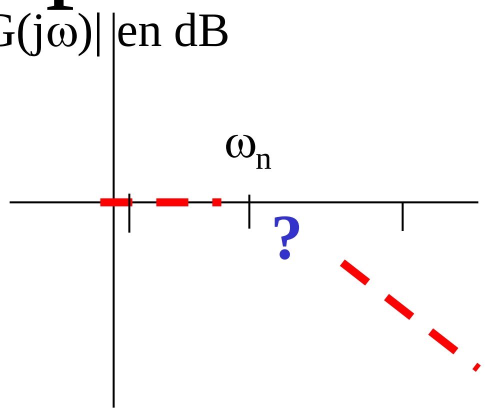
$$20\log\frac{1}{\left|1-\frac{\omega^2}{\omega_n^2}+j\frac{2\delta\omega}{\omega_n}\right|} = -20\log\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(\frac{2\delta\omega}{\omega_n}\right)^2}$$

$$\text{si}\ \ \omega\rightarrow 0 \qquad 20\log|.|\rightarrow 0$$

$$\text{si}\ \ \omega >> \omega_n \qquad 20\log|.|\rightarrow -20\log\frac{\omega^2}{\omega_n^2} = -40\log\omega+40\log\omega_n$$

straight line of slope - 40 dB passing through ($\omega = \omega_n$, 0 dB)

Bode: complex conjugate poles



Does it present a maximum?

$$20\log\frac{1}{\left|1-\frac{\omega^2}{\omega_n^2}+j\frac{2\delta\omega}{\omega_n}\right|}=-20\log\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(\frac{2\delta\omega}{\omega_n}\right)^2}$$

$$\frac{d}{d\omega}\left[\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(\frac{2\delta\omega}{\omega_n}\right)^2\right]=0 \qquad 2\left(1-\frac{\omega^2}{\omega_n^2}\right)\left(-\frac{2\omega}{\omega_n^2}\right)+\frac{8\delta^2\omega}{\omega_n^2}=0$$

$$-\left(\omega_n^2-\omega^2\right)+2\delta^2\omega_n^2=0 \qquad \omega_r=\omega_n\sqrt{1-2\delta^2}$$

if $\delta \le 0.707$ there will be a maximum in $|G(j\omega)|$ known as resonance peak M_r at the frequency $\omega_r \le \omega_n$

$$M_r=|G(j\omega_r)|=\frac{1}{2\delta\sqrt{1-\delta^2}}$$

Bode: complex conjugate poles

$$\frac{\omega_n^2}{s^2+2\delta\omega_n s+\omega_n^2} \rightarrow \frac{1}{\left(\frac{j\omega}{\omega_n}\right)^2+2\delta\frac{j\omega}{\omega_n}+1} = \frac{1}{1-\frac{\omega^2}{\omega_n^2}+j\frac{2\delta\omega}{\omega_n}}$$

$$\arg\frac{1}{1-\frac{\omega^2}{\omega_n^2}+j\frac{2\delta\omega}{\omega_n}} = -\operatorname{arctg}\left(\frac{\frac{2\delta\omega}{\omega_n}}{1-\frac{\omega^2}{\omega_n^2}}\right)$$

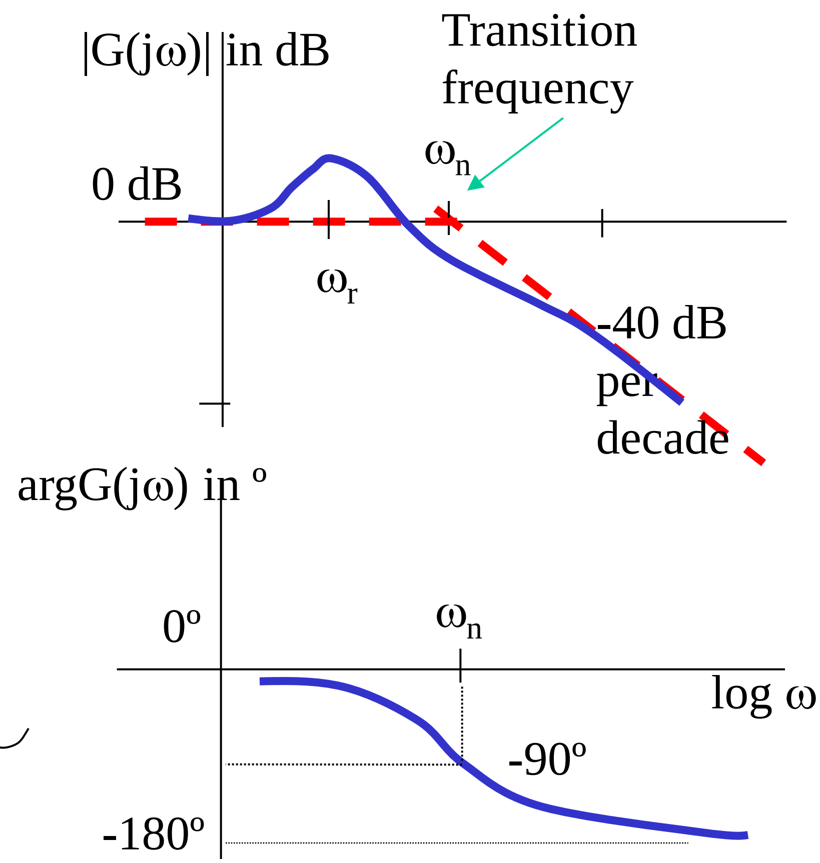
$$\text{if} \quad \omega \rightarrow 0 \qquad \phi \rightarrow 0$$

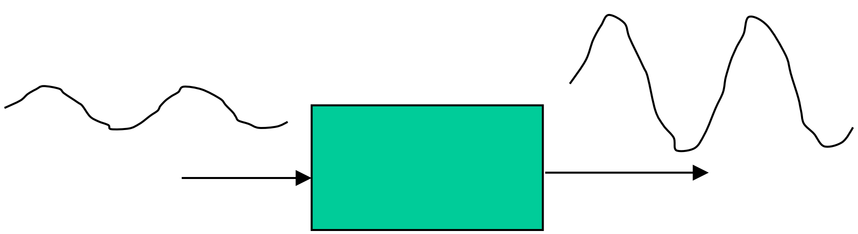
$$\text{if} \quad \omega = \omega_n \qquad \phi = -90^{\circ}$$

$$\text{if} \quad \omega \rightarrow \infty \qquad \phi \rightarrow -180^{\circ}$$

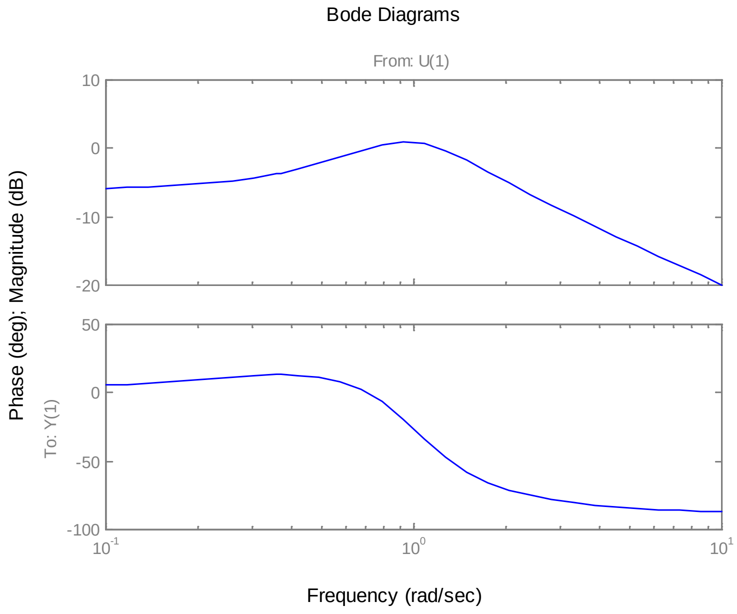
Case $\delta < 0.707$

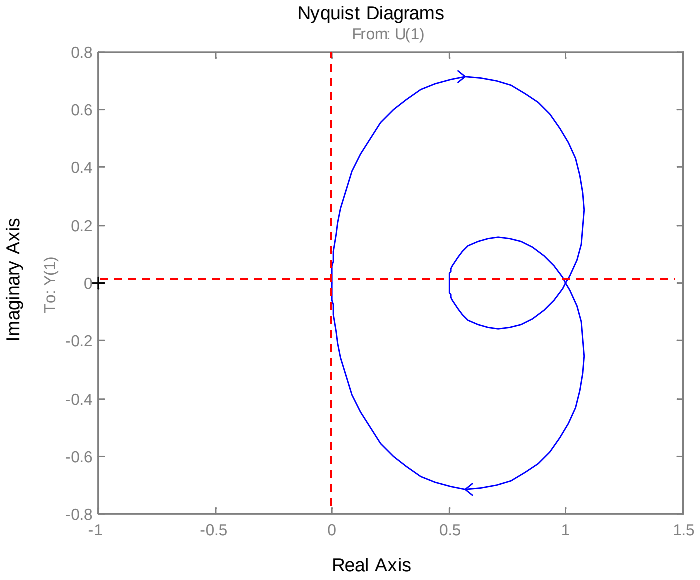
Resonance: The magnitude of the output signal is amplified for a range of frequencies and it is maximum at $\omega_{r,}$, increasing with decreasing δ





Example

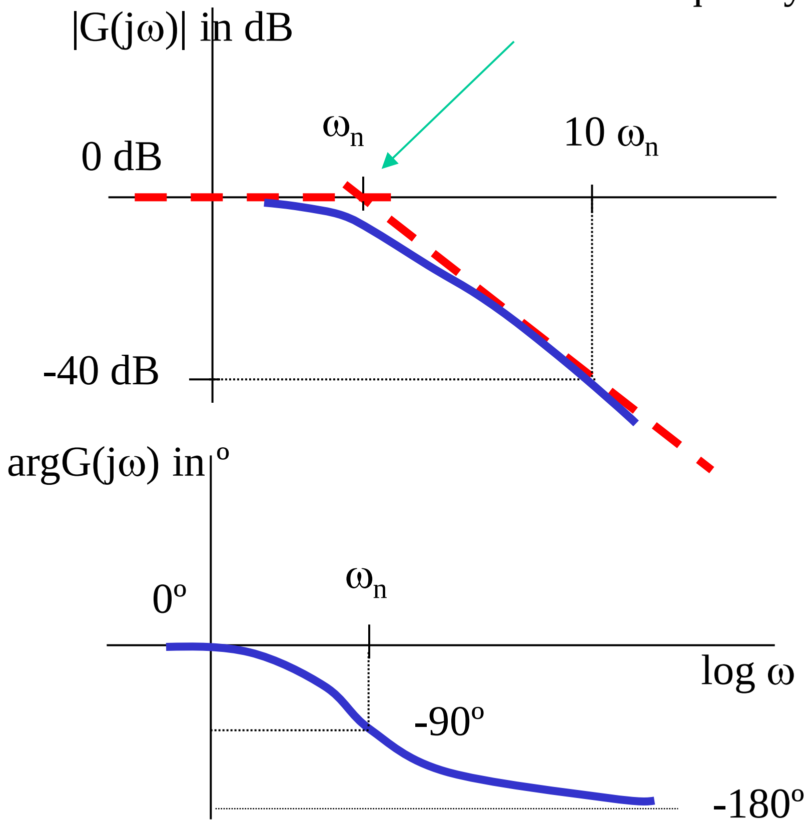

Bode Diagrams
From: U(1)
10
0
-10
-20
50
0
-50
-100
Phase (deg); Magnitude (dB)
To: Y(1)
10^{-1}
10^{0}
10^{1}
Frequency (rad/sec)


Nyquist Diagrams
From: U(1)
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
Imaginary Axis
To: Y(1)
-1
-0.5
0
0.5
1
1.5
Real Axis

Case $\delta > 0.707$

Transition frequency = natural frequency

There is no resonance. The attenuation is monotonously decreasing at a rate -40dB per decade for frequencies higher than ω_n

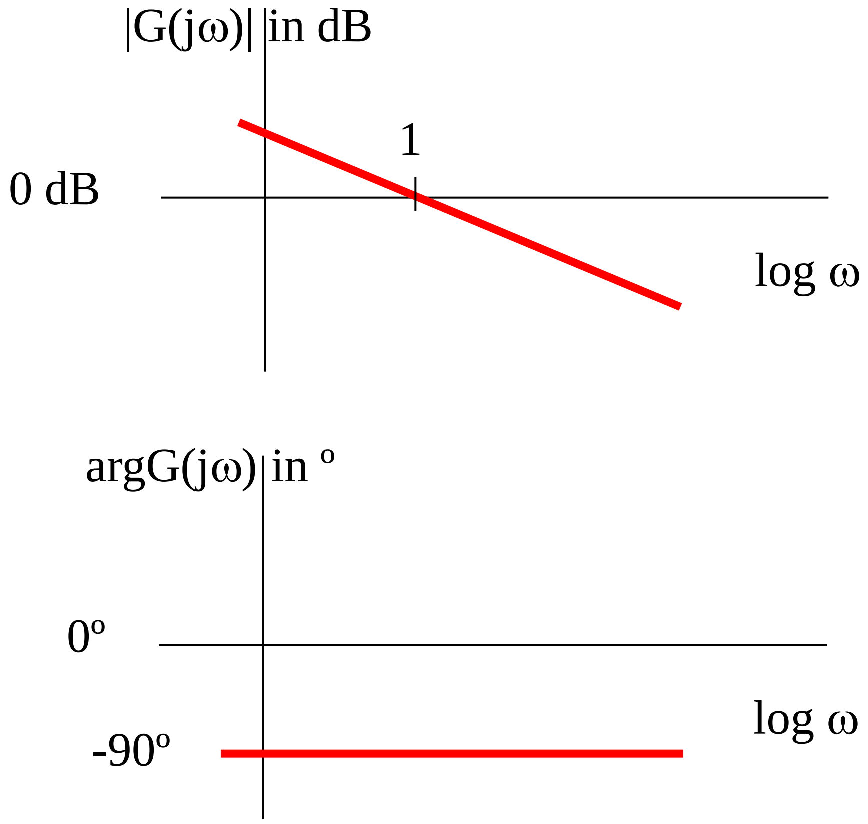


Bode: integrators

$$20\log\left|\frac{1}{j\omega}\right| = -20\log\omega$$

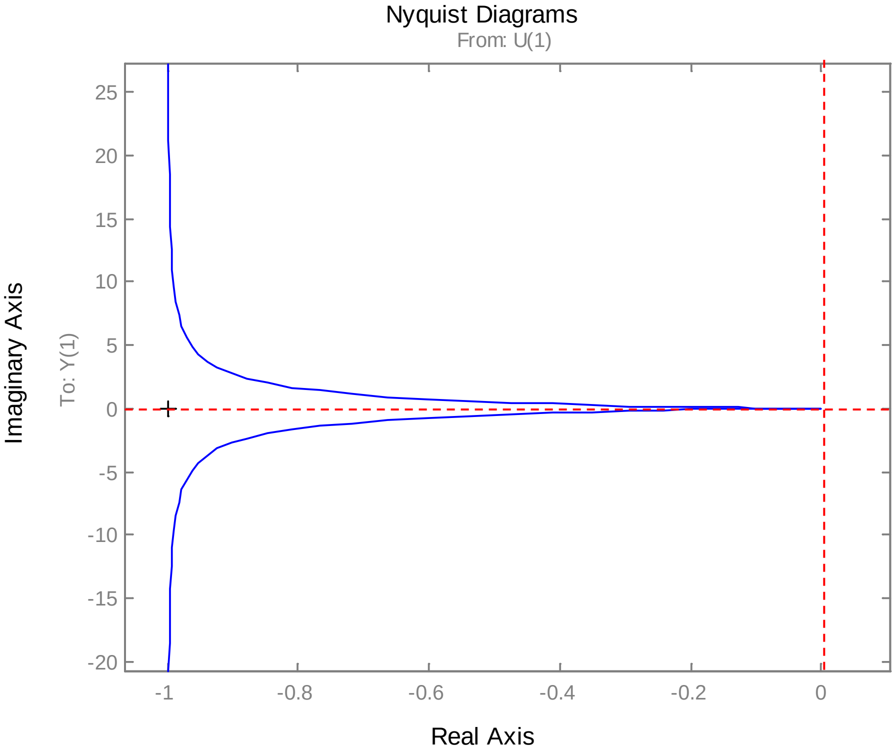
$$\arg\left(\frac{1}{j\omega}\right) = -90^{o}$$

Straight line of slope -20 dB
passing through (ω =1, 0 dB)



First order plus integrator

$$\frac{1}{s(s+1)}$$



Delays

- When delays are present in a transfer function, it is difficult to apply certain analysis techniques such as the root locus method
- This techniques requires to approximate the delay by a set of poles and zeros using the Pade approximation method

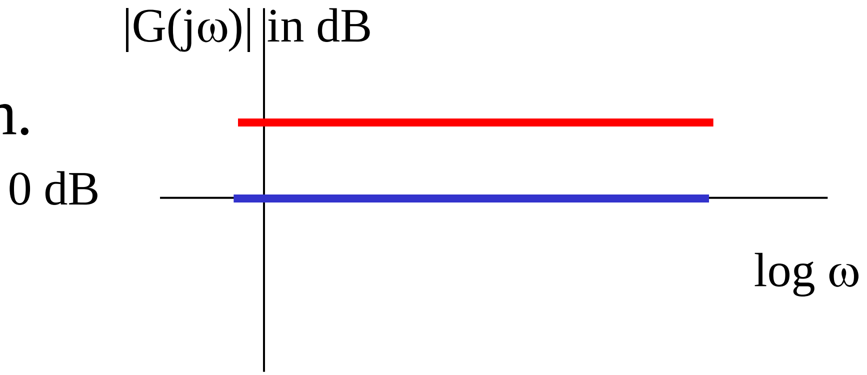
$$\frac{e^{-2s}}{s+1} \approx \frac{(s^2 - 3s + 3)}{(s^2 + 3s + 3)(s+1)}$$

- Nevertheless, in the frequency domain, the analysis of a systems with delay does not imply any special difficulty

Bode: K, delay

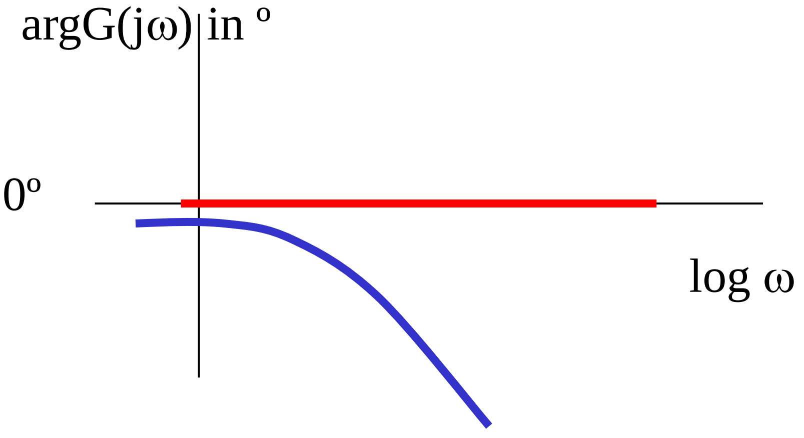
$$20\log|K|$$ Is a constant term.

$$\arg(K) = 0 \text{ or } -\pi$$



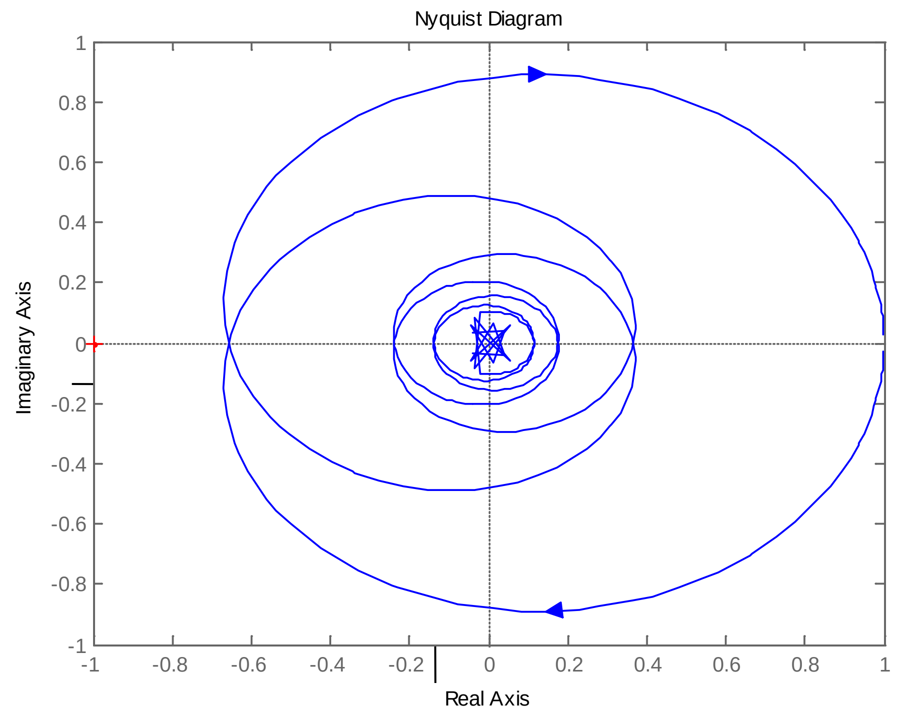
$$20\log\left|e^{-j\omega d}\right| = 20\log 1 = 0$$

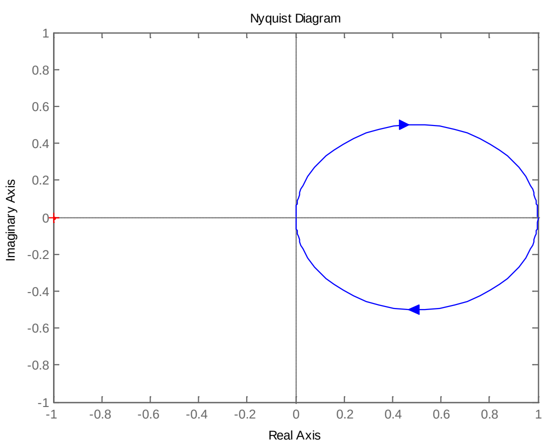
$$\arg(e^{-j\omega d}) = -\omega d$$



First order plus delay

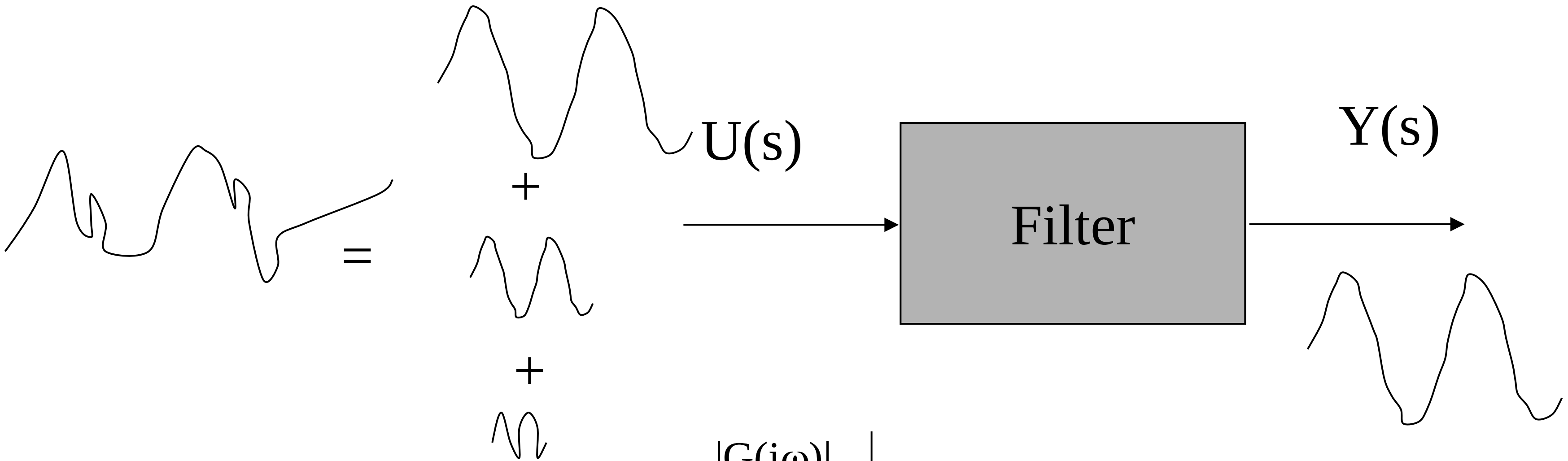
$$\frac{e^{-2s}}{s+1}$$

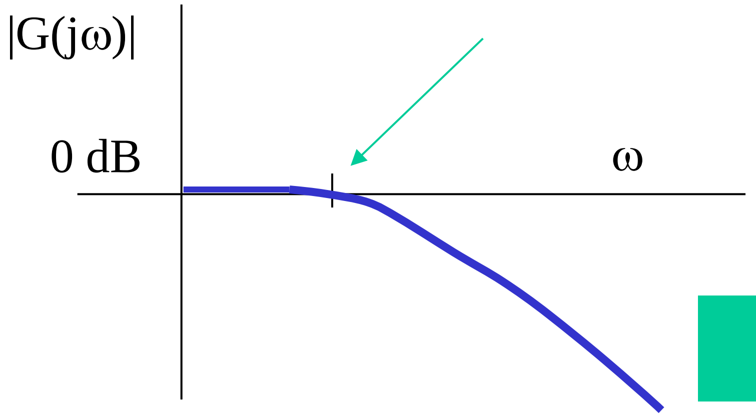




$$\frac{1}{s+1}$$

Filters



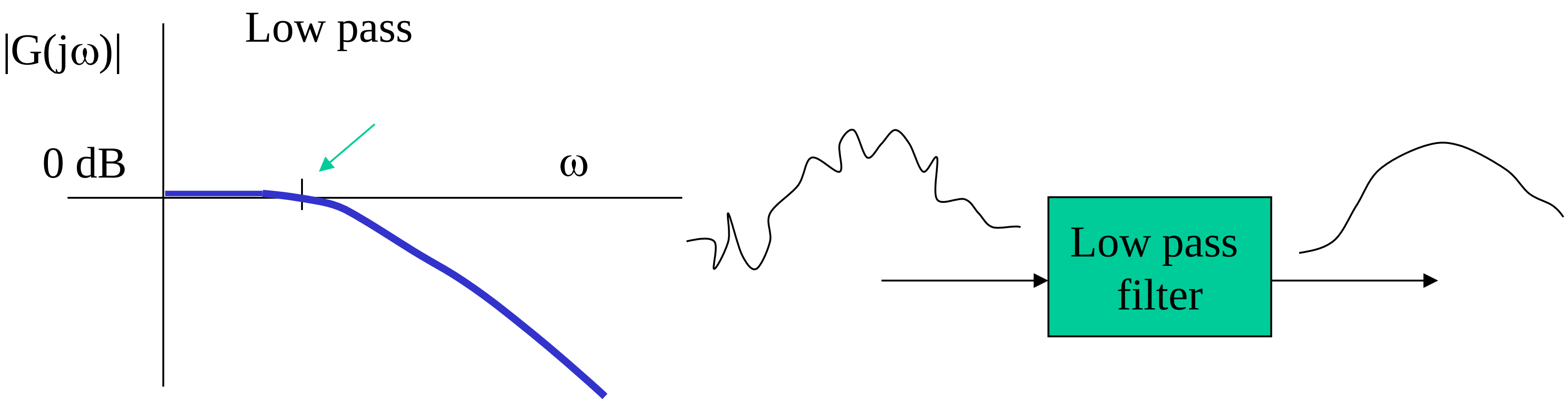


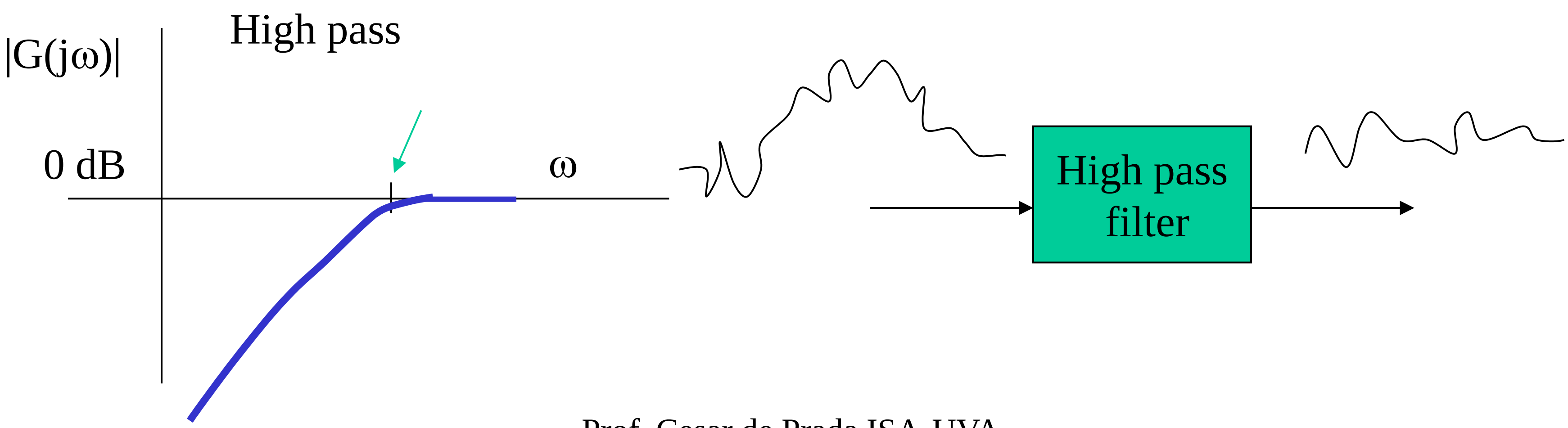
A filter is a device that eliminates a range of undesired frequencies in a signal

Filters

It adds a delay in the signal!

Types of filters

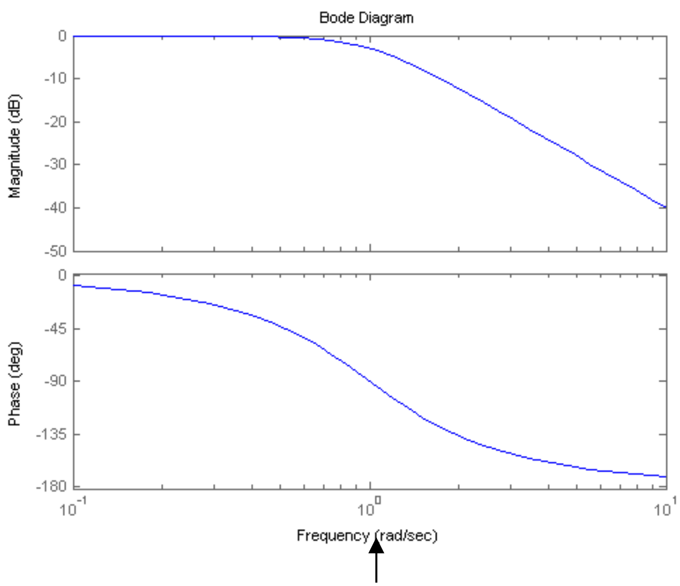


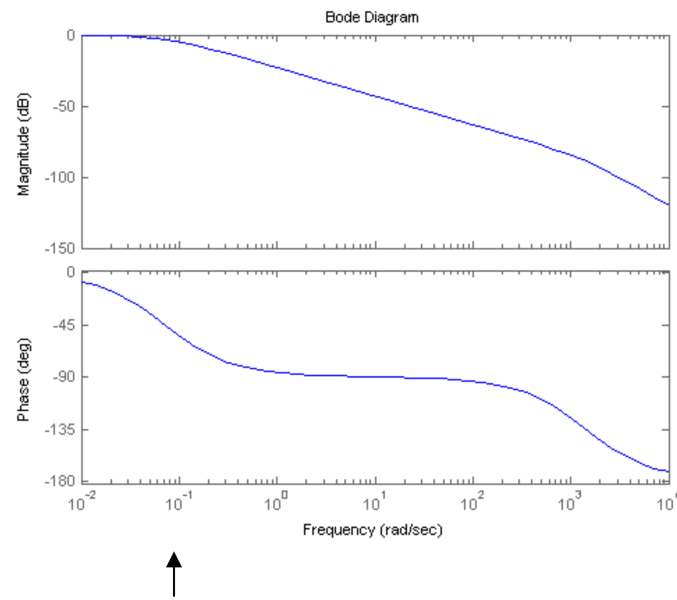


Example: 2nd order Butterworth filter

Prototype 2nd order Butterworth filter

$$\frac{1}{s^2 + 1.42s + 1} \qquad s \to \frac{s}{0.1} \qquad \frac{100}{s^2 + 1420s + 100}$$

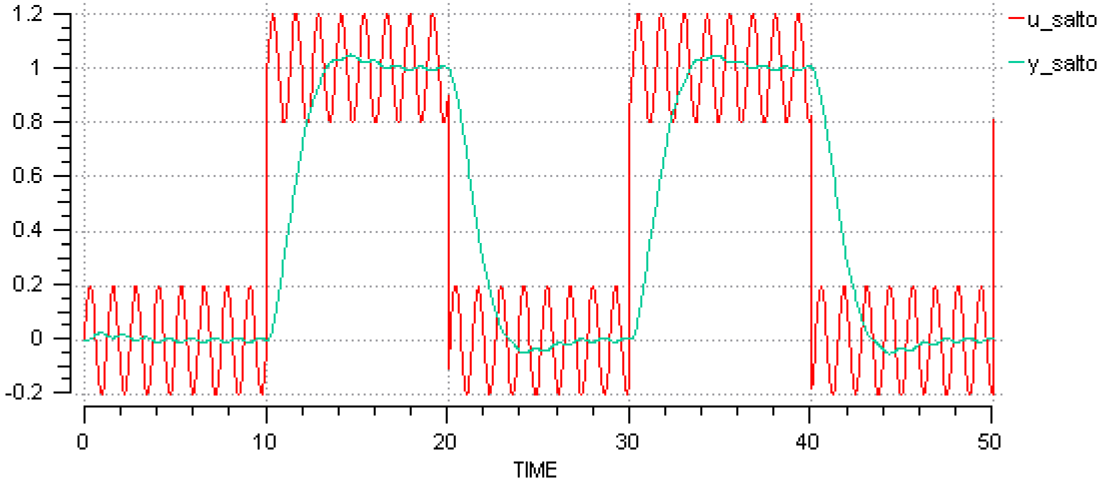


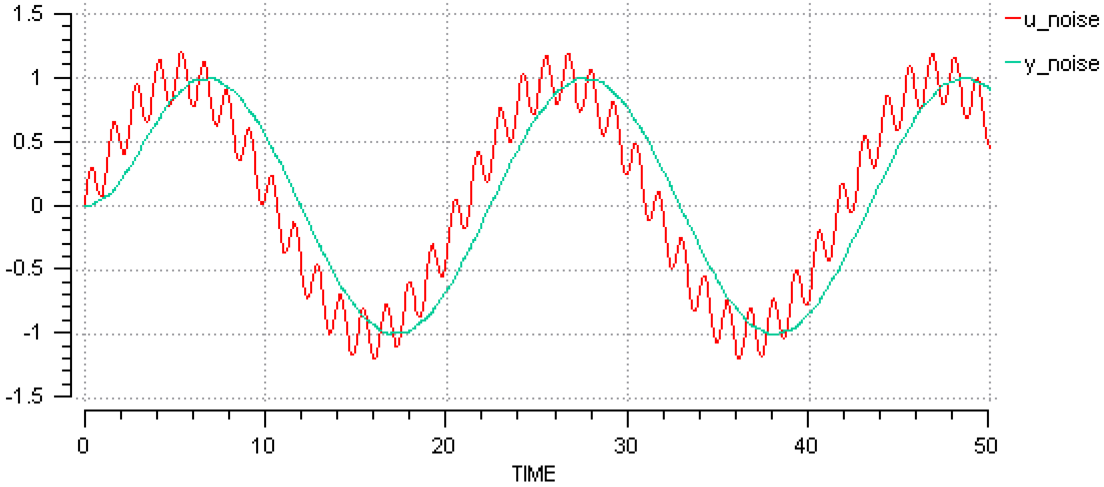


$\omega_c = 1$ $\omega_c = 0.1$

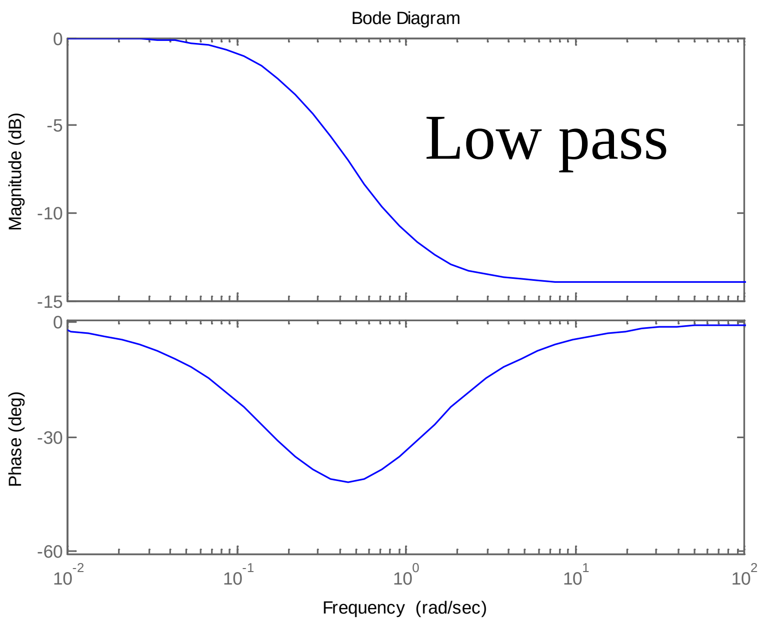
Example: 2nd order Butterworth filter

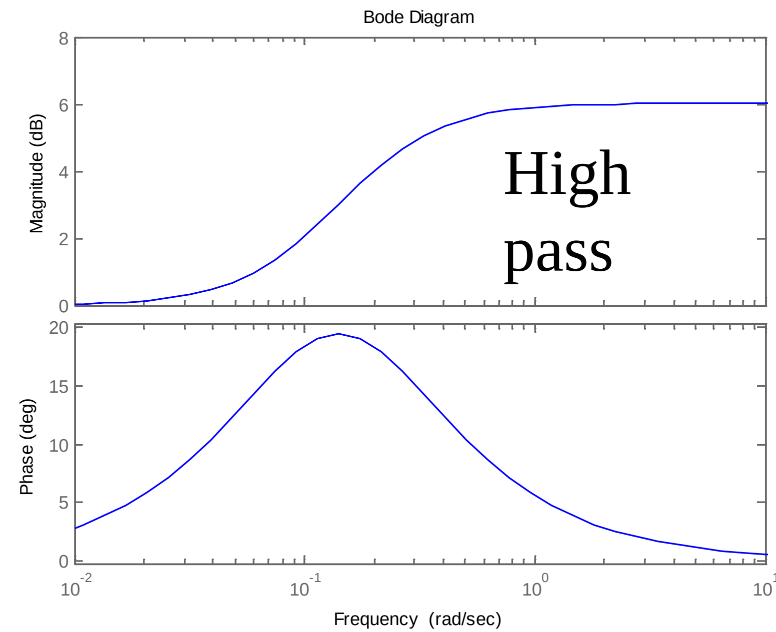
$w_c = 1$





Lead/Lag Zero/pole

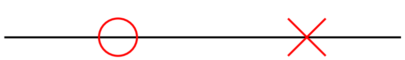




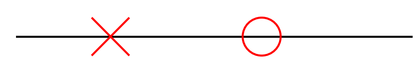
$$\frac{s+1}{5s+1}$$

The zero location determines the systems response at high frequencies

$$\frac{10s+1}{5s+1}$$

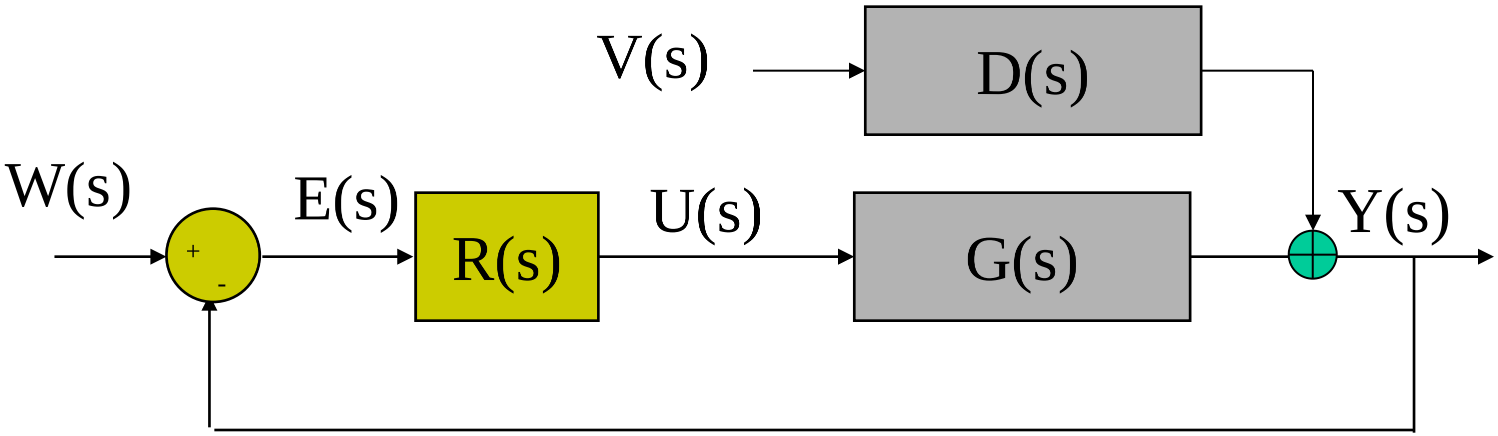


Pole to the right



Pole to the left

Closed loop frequency response

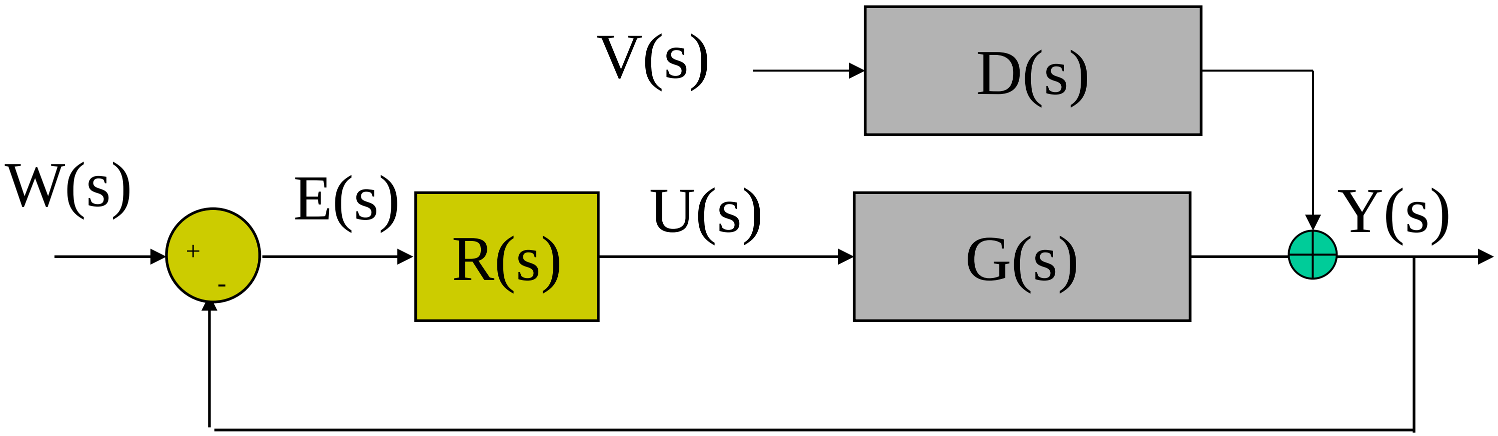


$$Y(s) = \frac{G(s)R(s)}{1+G(s)R(s)} W(s) + \frac{D(s)}{1+G(s)R(s)} V(s)$$

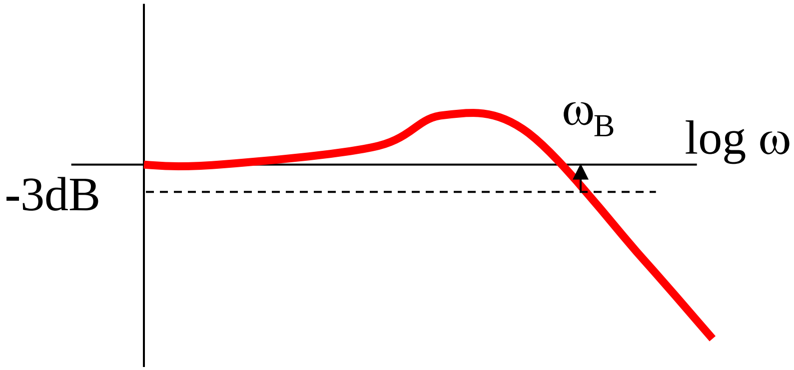
The same type of diagrams can be used to study the closed loop response

$$\frac{G(j\omega)R(j\omega)}{1+G(j\omega)R(j\omega)} \qquad \frac{D(j\omega)}{1+G(j\omega)R(j\omega)}$$

Closed loop frequency response



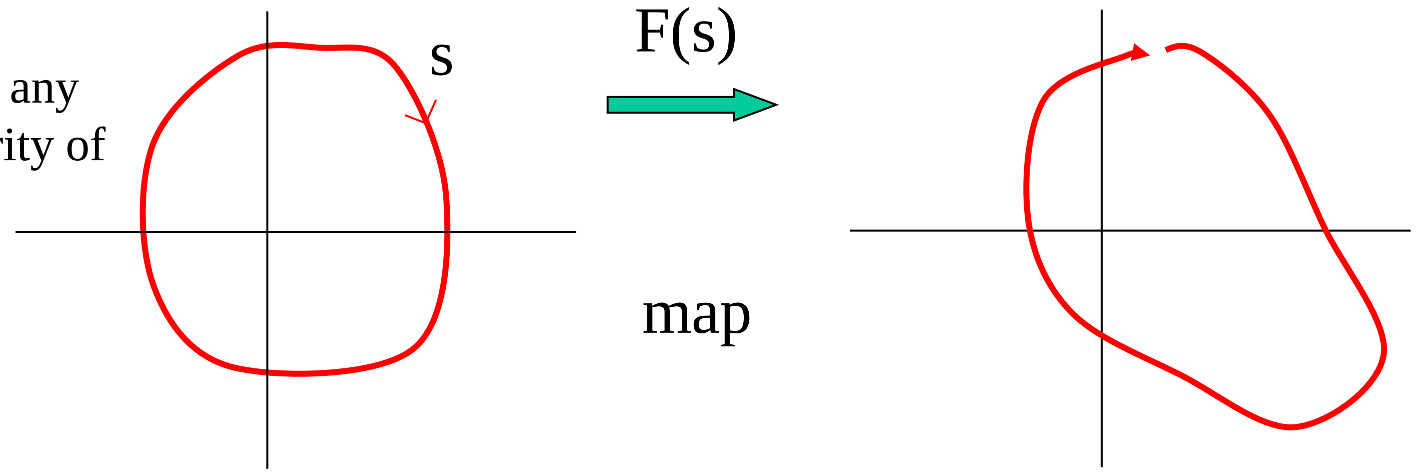
$$\frac{G(j\omega)R(j\omega)}{1+G(j\omega)R(j\omega)} \qquad \frac{D(j\omega)}{1+G(j\omega)R(j\omega)}$$



The bandwidth gives an indication of the speed of response and disturbance rejection properties

Cauchy's argument principle

Closed contour not passing through any singularity of F(s)



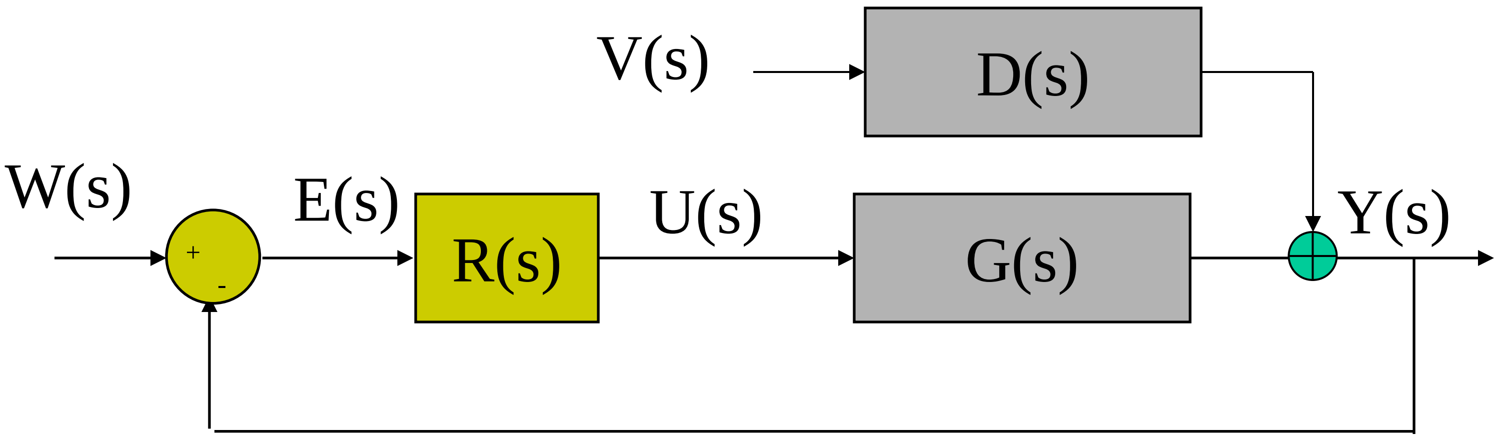
P number of poles of F(s) within the contour

Z number of zeros of F(s) within the contour

N number of times that F(s) encircles the origin clockwise

$$N = Z - P$$

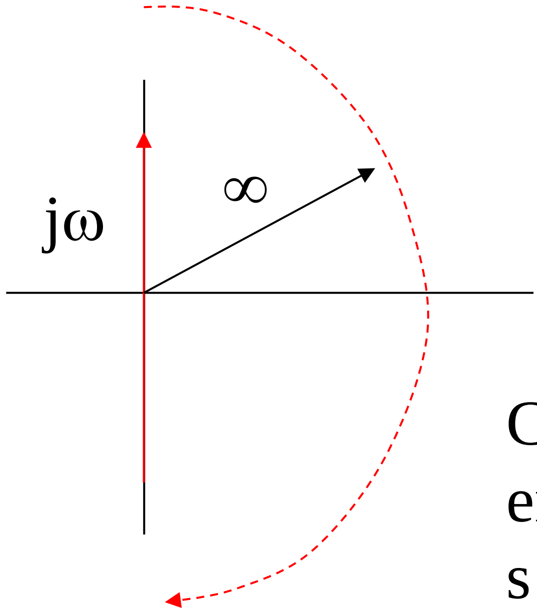
Closed loop stability

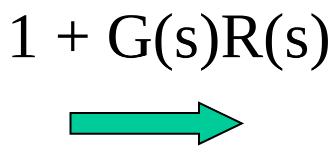


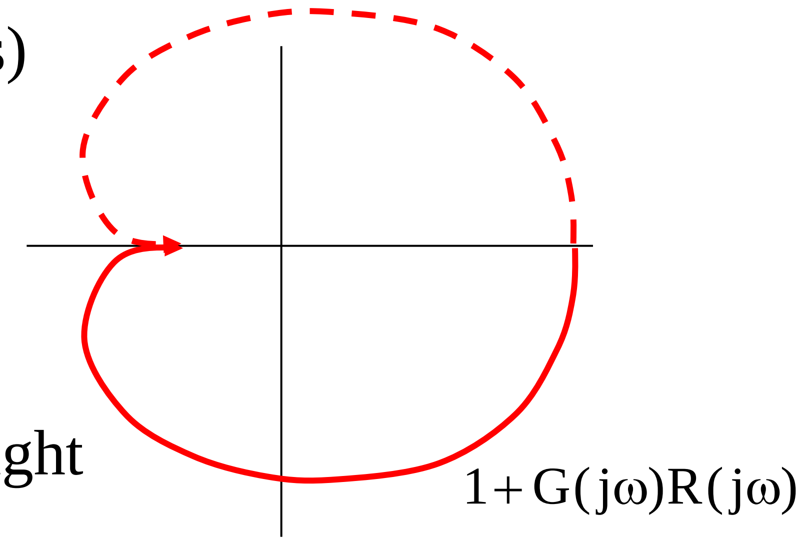
$$Y(s) = \frac{G(s)R(s)}{1+G(s)R(s)} W(s) + \frac{D(s)}{1+G(s)R(s)} V(s)$$

How many roots of $1 + G(s)R(s) = 0$ are positive?

The Nyquist contour







Contour that encloses the right s half plane

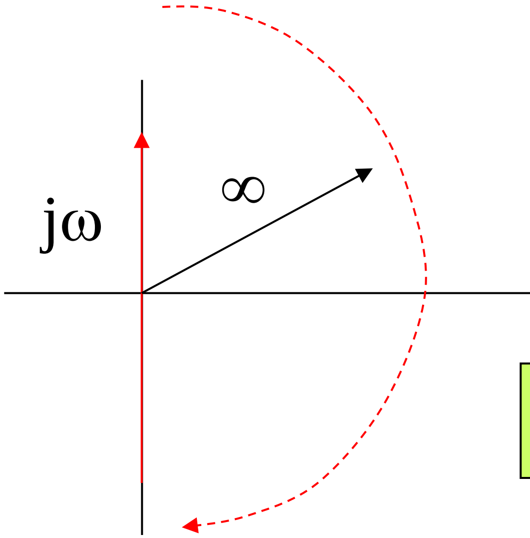
$$1 + GR = 1 + \frac{\text{Num}}{\text{Den}} = \frac{\text{Den} + \text{Num}}{\text{Den}}$$

Poles of 1+GR = poles of GR

P = number of unstable poles of GR

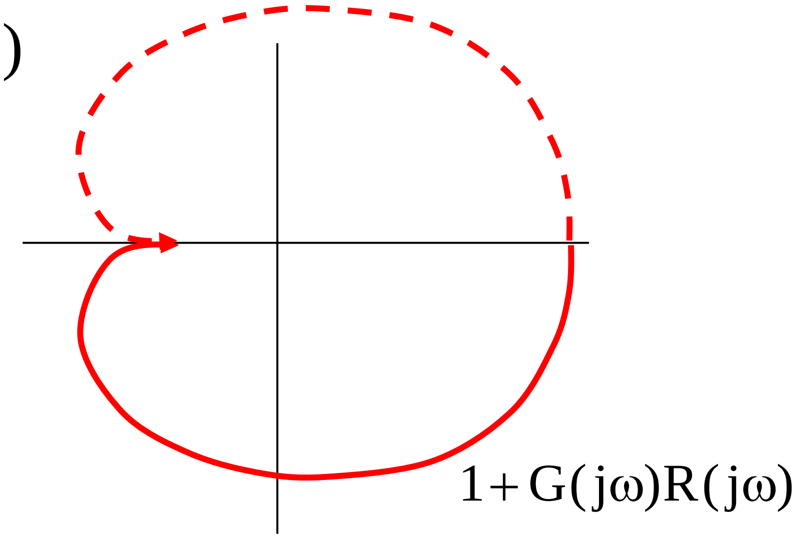
Z = number of zeros of 1+GR in the right s half plane

Nyquist criterion



1 + G(s)R(s)

N = Z - P



P = number of unstable poles of GR

Z = number of zeros of 1+GR in the right s half plane

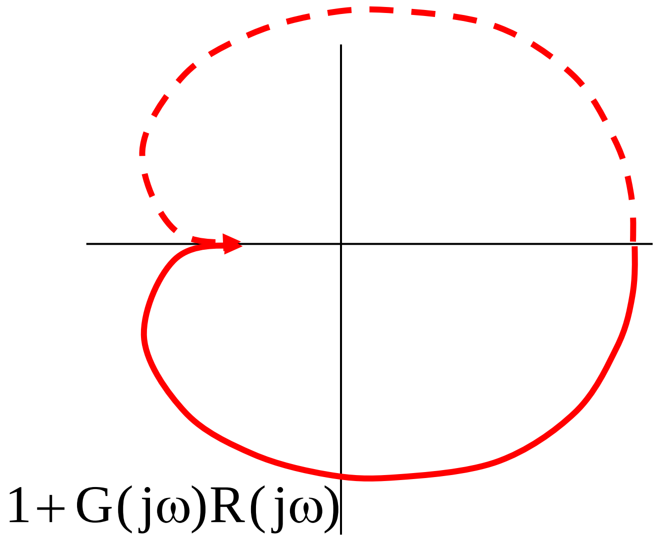
N = number of times that 1+G(jω)R(jω) encircles the origin clockwise

Closed loop stability implies Z = 0, so:

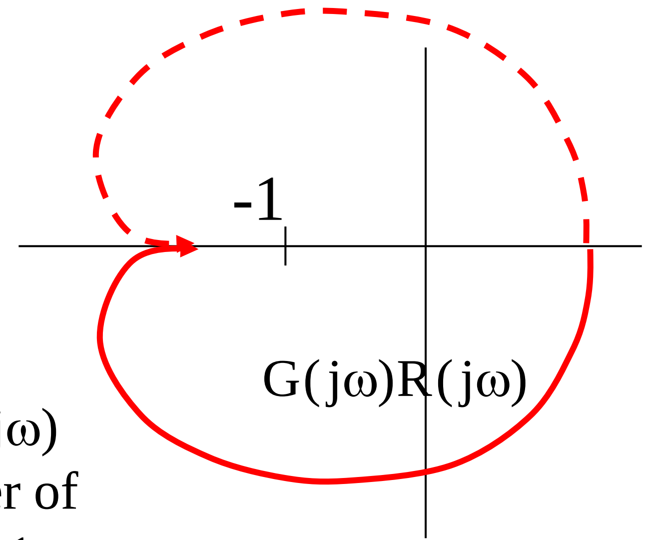
N = - P

Nyquist criterion

Nyquist Criterion



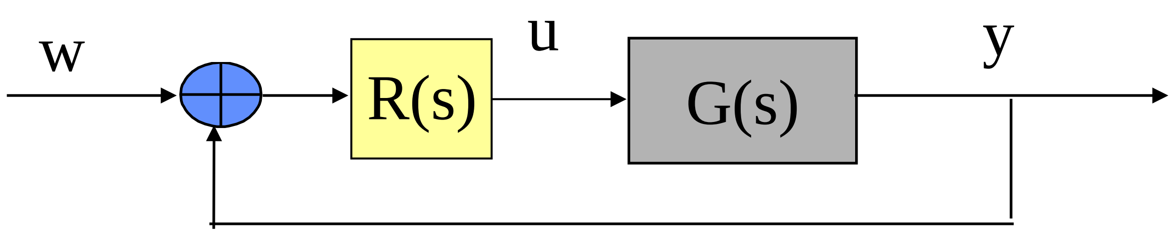
The number of encirclements to the origen of 1+G(jω)R(jω) is equal to the number of encirclements to the -1 point of G(jω)R(jω)



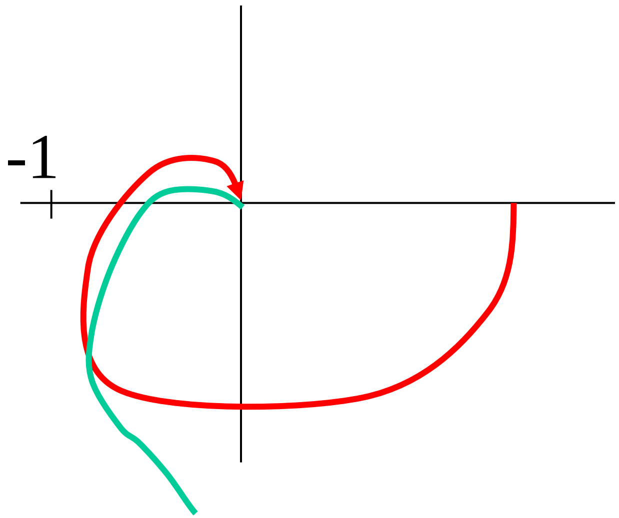
If the process is open loop stable, then P = 0, and the closed loop stability requires that the Nyquist diagram does not encircle the (-1,0) point

Sysquake

Robustness

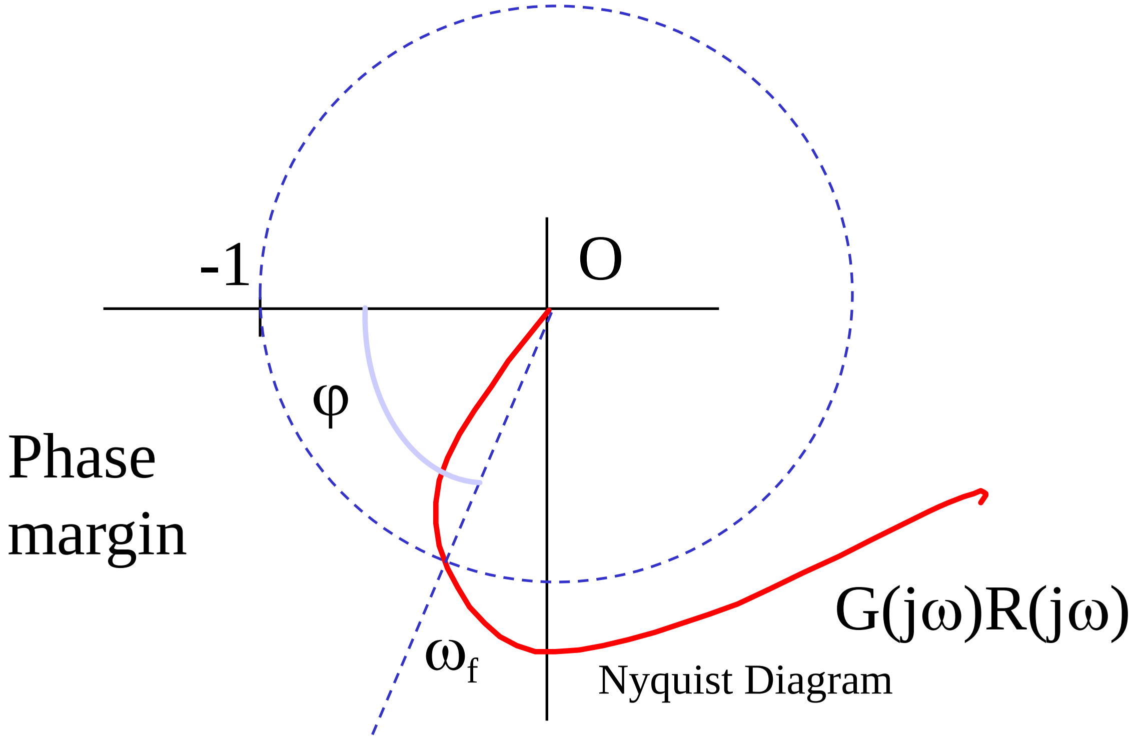


If the model differs from the process, or the process or the controller tuning change, will the closed loop system be stable?



How far is the closed loop system from becoming unstable?

Phase margin PM

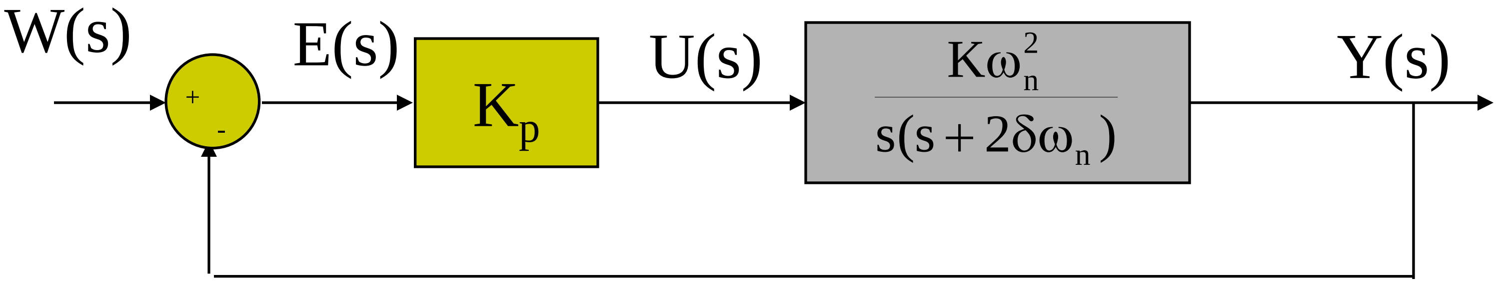


The PM indicates how far the closed loop system is from unstability with respect to phase shifts. Phase margin must be positive in a closed loop stable system

ω_f highest frequency at which $\quad |G(j\omega_f)R(j\omega_f)| = 1$

φ angle that verifies $\quad \arg(G(j\omega_f)R(j\omega_f)) = -\pi + \varphi$

Example Phase margin, 2nd order

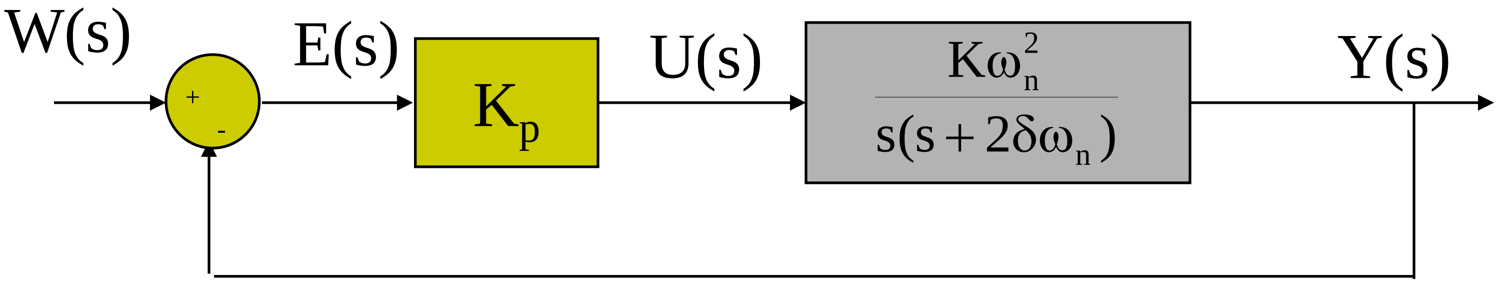


In closed loop:

$$\frac{G(s)K_p}{1+G(s)K_p} = \frac{K_p K\omega_n^2}{s^2 + 2\delta\omega_n s + K_p K\omega_n^2}$$

Which is the PM of this system?
Which is the relation of the PM and the closed loop dynamics?

Phase margin, 2nd order



If the phase margin is obtained at the frequency ω_f :

$$\left|\frac{KK_p\omega_n^2}{s(s+2\delta\omega_n)}\right|_{s=j\omega_f} = 1 \quad \Rightarrow \quad KK_p\omega_n^2 = \sqrt{(-\omega_f^2)^2 + 4\delta^2\omega_n^2\omega_f^2}$$

$$K^2K_p^2\omega_n^4 = \omega_f^4 + 4\delta^2\omega_n^2\omega_f^2$$

$$\omega_f^4 + 4\delta^2\omega_n^2\omega_f^2 - K^2K_p^2\omega_n^4 = 0$$

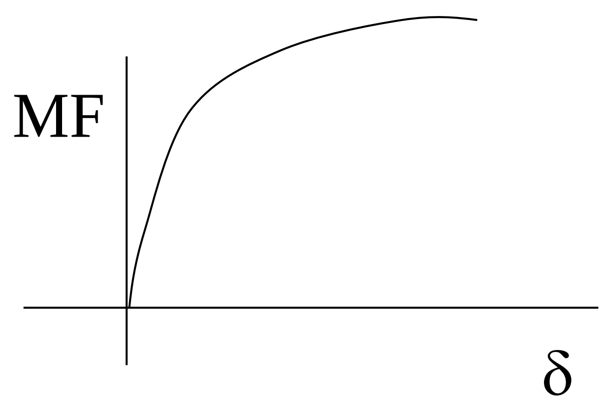
$$\omega_f^2 = \frac{-4\delta^2\omega_n^2 \pm \sqrt{(4\delta^2\omega_n^2)^2 - 4K^2K_p^2\omega_n^4}}{2} = \omega_n^2(-2\delta^2 \pm \sqrt{4\delta^4 - K^2K_p^2})$$

Phase margin, 2nd order

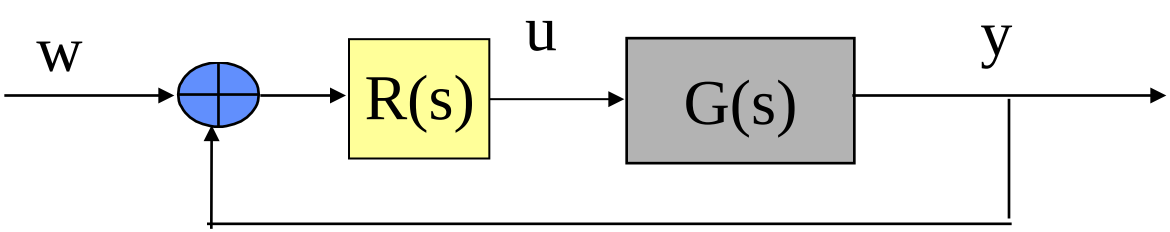
$$\omega_f^2 = \omega_n^2(-2\delta^2 \pm \sqrt{4\delta^4 - K^2K_p^2})$$

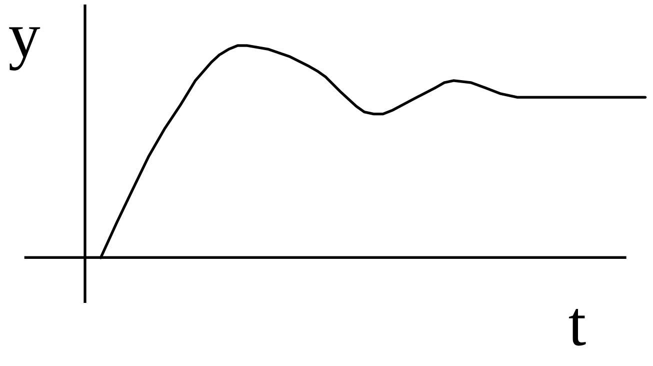
$$\varphi = \pi + \arg \left. \frac{KK_p\omega_n^2}{s(s+2\delta\omega_n)} \right|_{s=j\omega_f} = \pi - \frac{\pi}{2} - \text{arctg}\left(\frac{\omega_f}{2\delta\omega_n}\right) = \frac{\pi}{2} - \text{arctg}\left(\frac{\sqrt{-2\delta^2 \pm \sqrt{4\delta^4 - K^2K_p^2}}}{2\delta}\right)$$

There is a direct relation between the phase margin φ and the damping δ in a second order system. For higher order systems the relation is only approximate



Phase margin





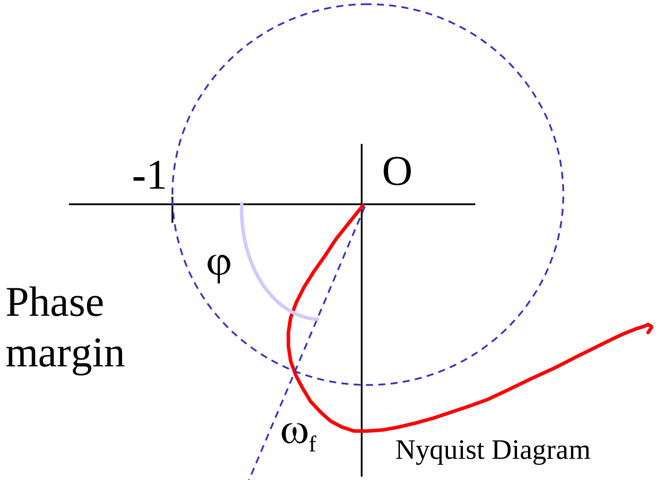
segundoorden

The phase margin φ is related to the overshoot and the stability. Systems with more overshoot tend to be less robust.

PM should be greater than 30º and ideally ~ 55º

The frequency ω_f is related to the speed of response

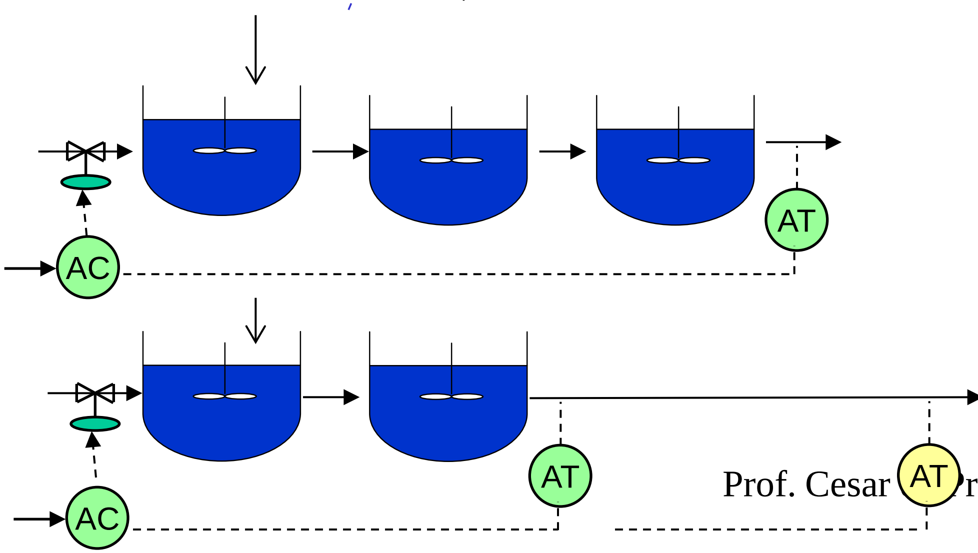
What effects tend to decrease the phase margin?



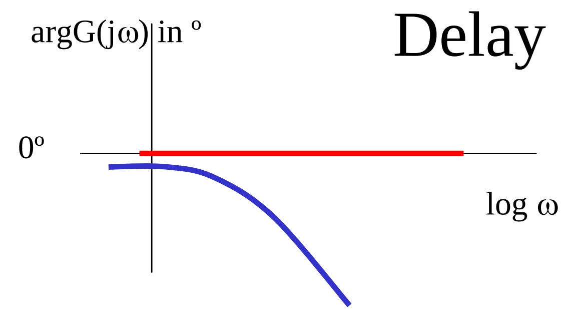
Those that increase the phase shift of G(s)R(s). In particular:

Adding more poles to the process

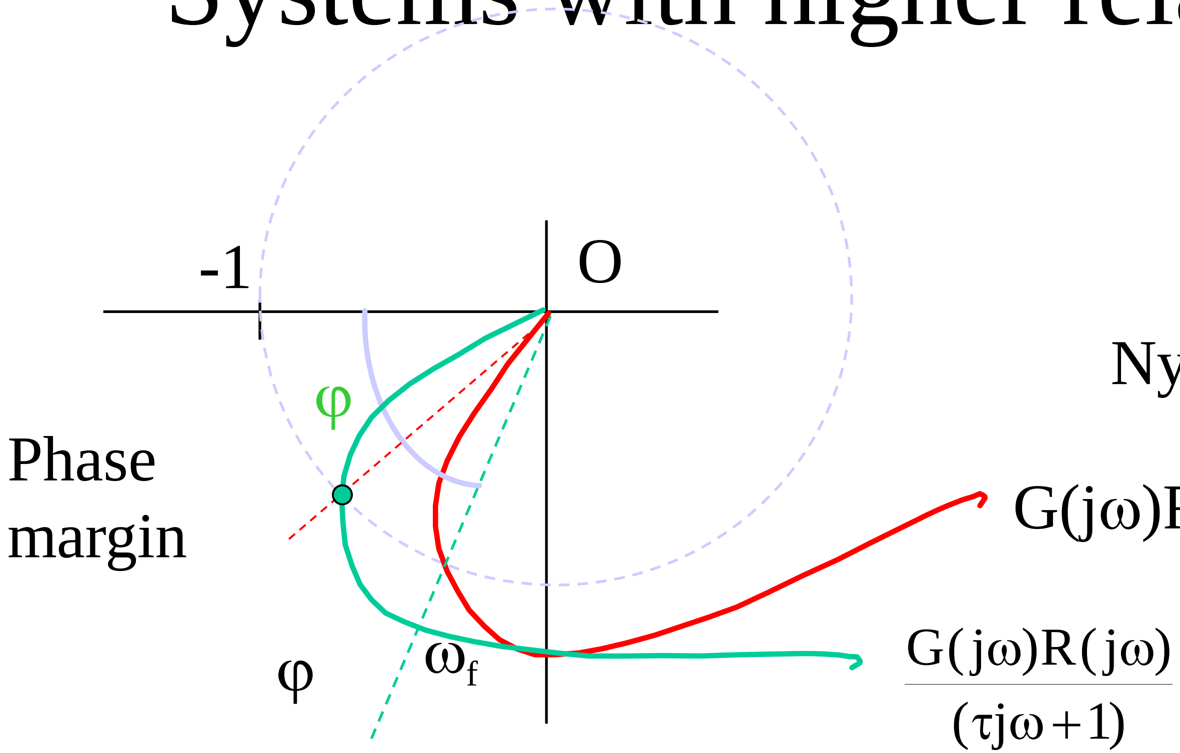
Increasing the process delays



Delay



Systems with higher relative order

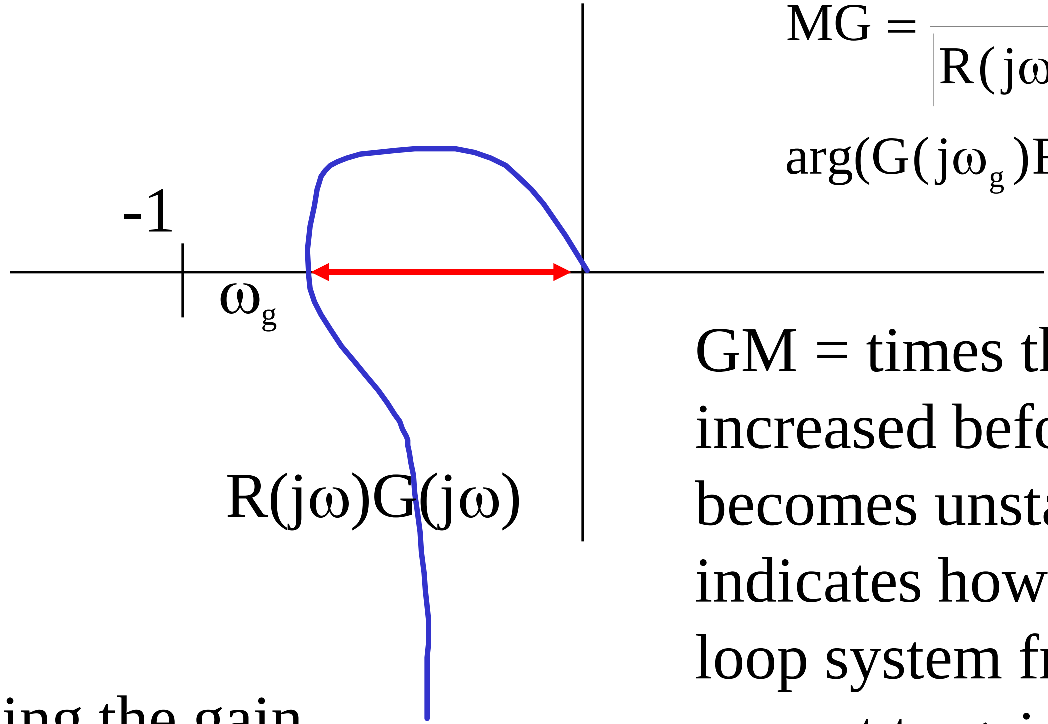


Nyquist Diagram

Systems with additional poles (resulting from adding a filter, etc.) are more difficult to control (less robust)

Increasing the excess of poles over the zeros decreases the phase margin

Gain Margin



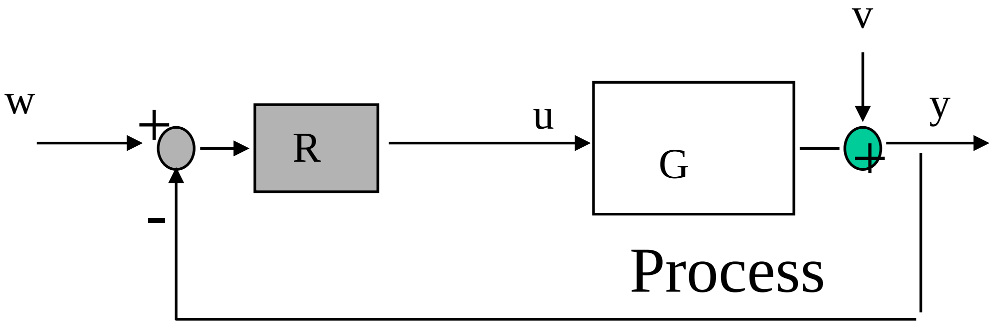
$$MG = \frac{1}{|R(j\omega_g)G(j\omega_g)|}$$

$$\arg(G(j\omega_g)R(j\omega_g)) = -\pi$$

GM = times the gain can be increased before the closed loop becomes unstable. The GM indicates how far is the closed loop system from unstability with respect to gain changes. GM must be greater than 1 in a closed loop stable system

Increasing the gain in the controller or the process will reduce the GM

Transfer functions



$$y = \frac{GR}{1+GR} w + \frac{1}{1+GR} v$$

S_{wy} S_{vy}

$$u = \frac{R}{1+GR} w - \frac{R}{1+GR} v$$

S_{wu} S_{vu}

$$y = \frac{G}{1/R+G} w + \frac{1}{1+GR} v$$

$$\text{if} \quad R \to \infty \quad y \to w + 0.v$$

Working with high gains can, according to this expression, improve SP following and disturbance rejection, but u will increase and also the stability and robustness will decrease….

Disturbance rejection

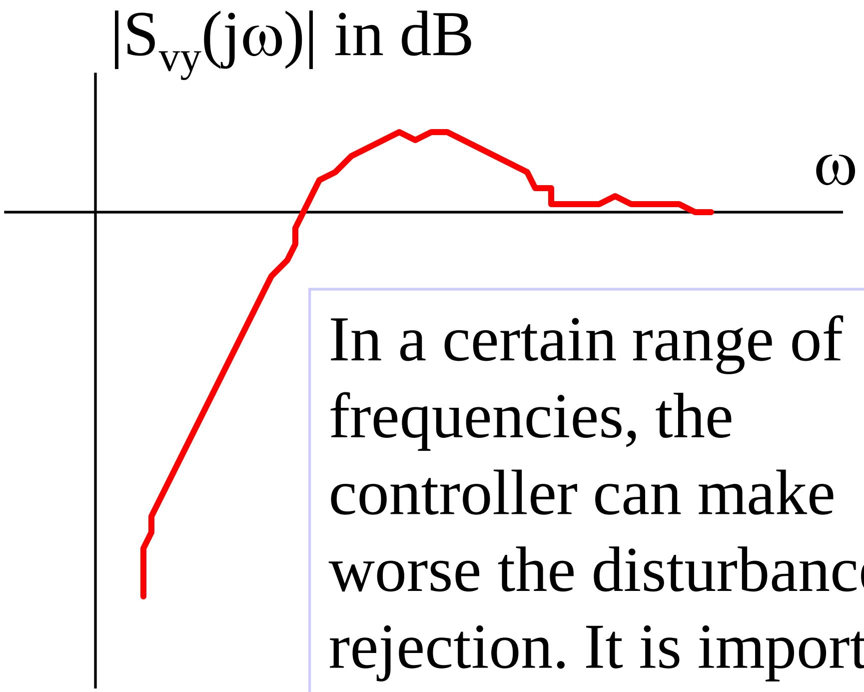
$$S_{vy} = \frac{1}{1+GR} =$$

$$= \frac{1}{1+G(j\omega)R(j\omega)}$$

if R have integral action

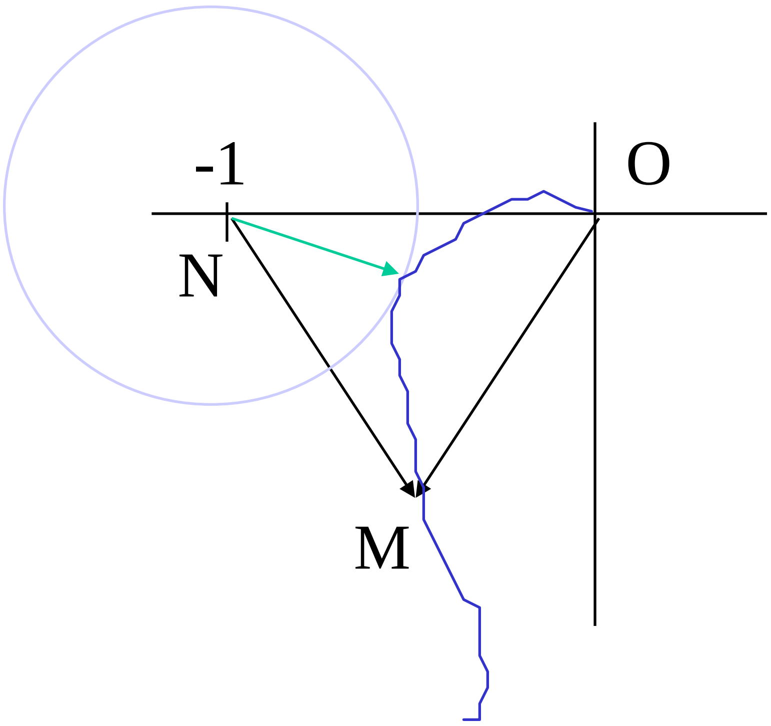
if $\omega \to 0$ then $S_{vy} \to 0$

if $\omega \to \infty$ then $S_{vy} \to 1$



In a certain range of frequencies, the controller can make worse the disturbance rejection. It is important to minimize the maximum $|S_{vy}(j\omega)|$

Modulus margin



Nyquist Diagram

$$\overline{-1} + \overline{NM} = \overline{OM} = G(j\omega)R(j\omega)$$

$$\left|\overline{NM}\right| = |1 + GR| = \left|S_{vy}^{-1}\right|$$

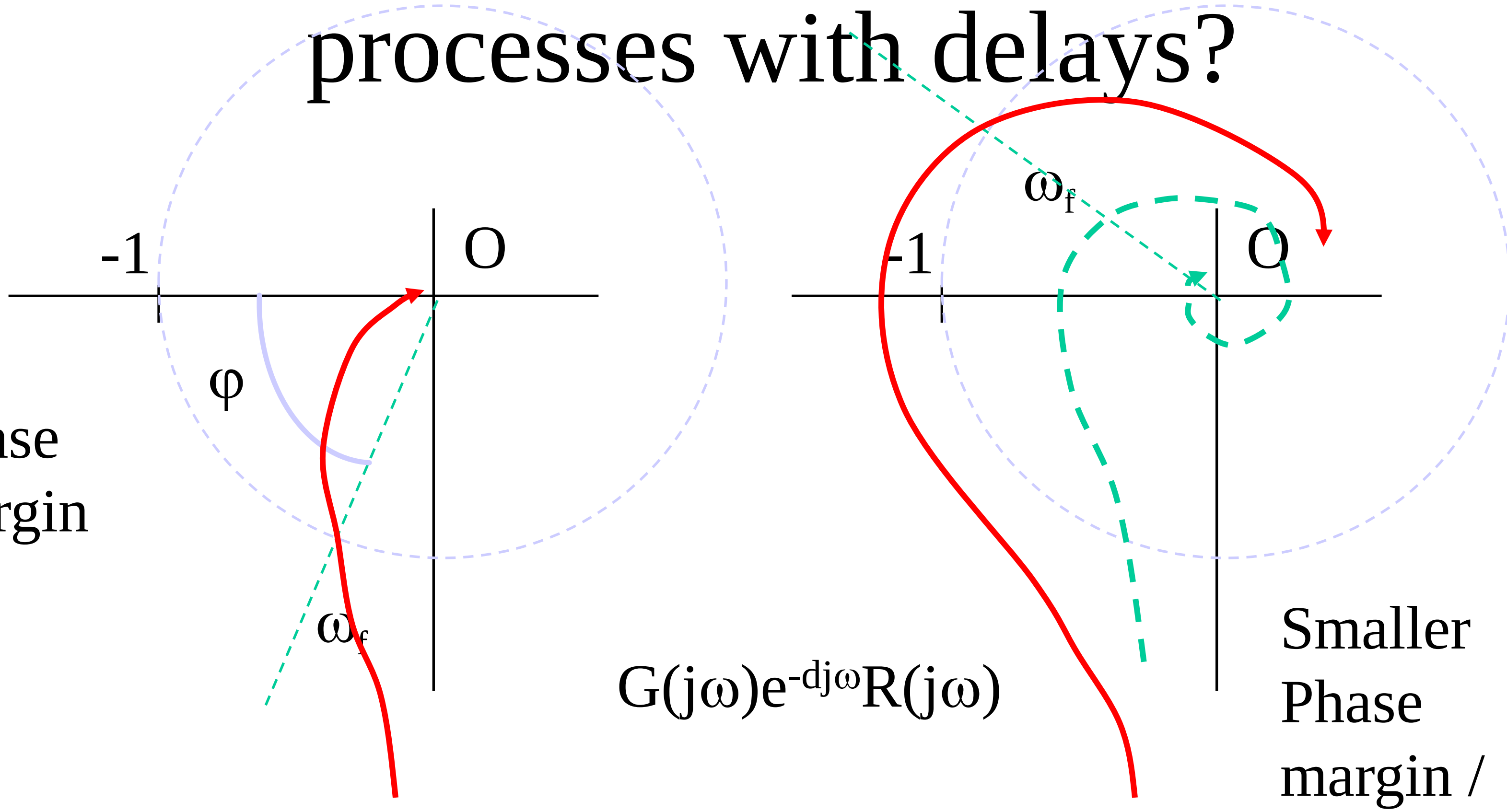
Modulus margin = min |NM|

$$\min |NM| = (\max \left|S_{vy}(j\omega)\right|)^{-1}$$

$$= \left\|S_{vy}(j\omega)\right\|_{\infty}^{-1}$$

Increasing the modulus margin, improves the disturbance rejection

Why is it difficult to control processes with delays?



Phase margin

G(jω)R(jω)

Smaller Phase margin / Modulus margin

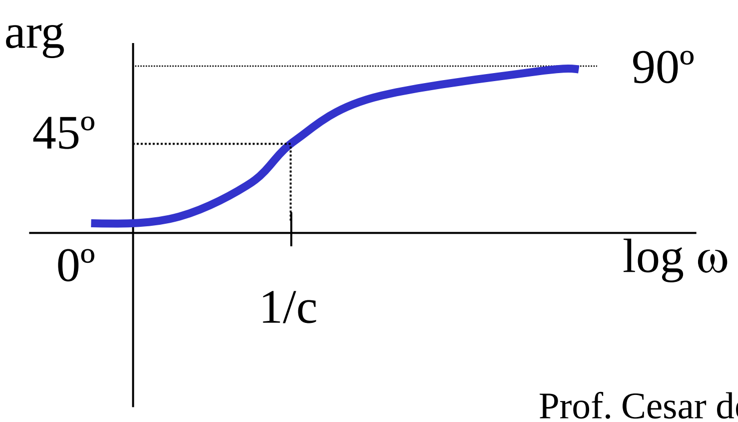
Non-minimum phase systems

$$G(s) = \frac{Ke^{-ds}(cs \pm 1)(\ldots)}{s(\tau s + 1)(\ldots)} \qquad 20\log\left|\frac{Ke^{-dj\omega}(cj\omega \pm 1)(\ldots)}{j\omega(\tau j\omega + 1)(\ldots)}\right|$$

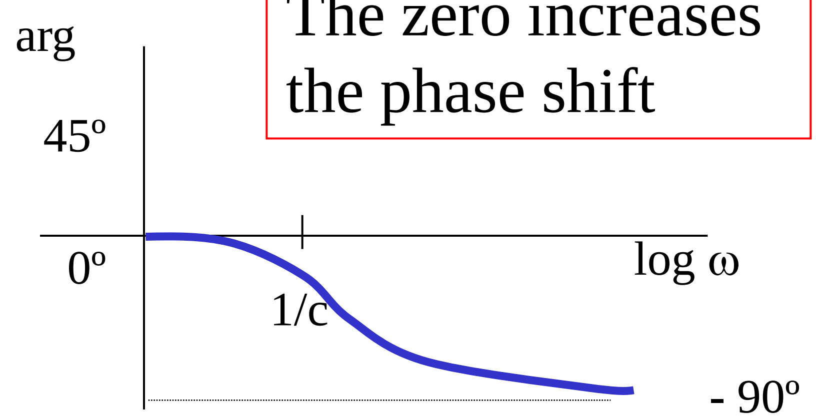
Modulus is not changed

$$\arg(G(j\omega)) = \arg(K) + \arg(e^{-j\omega d}) + \arg(cj\omega + 1) + \ldots + \arg(1/j\omega) + \arg(1/(\tau j\omega + 1)) + \ldots$$

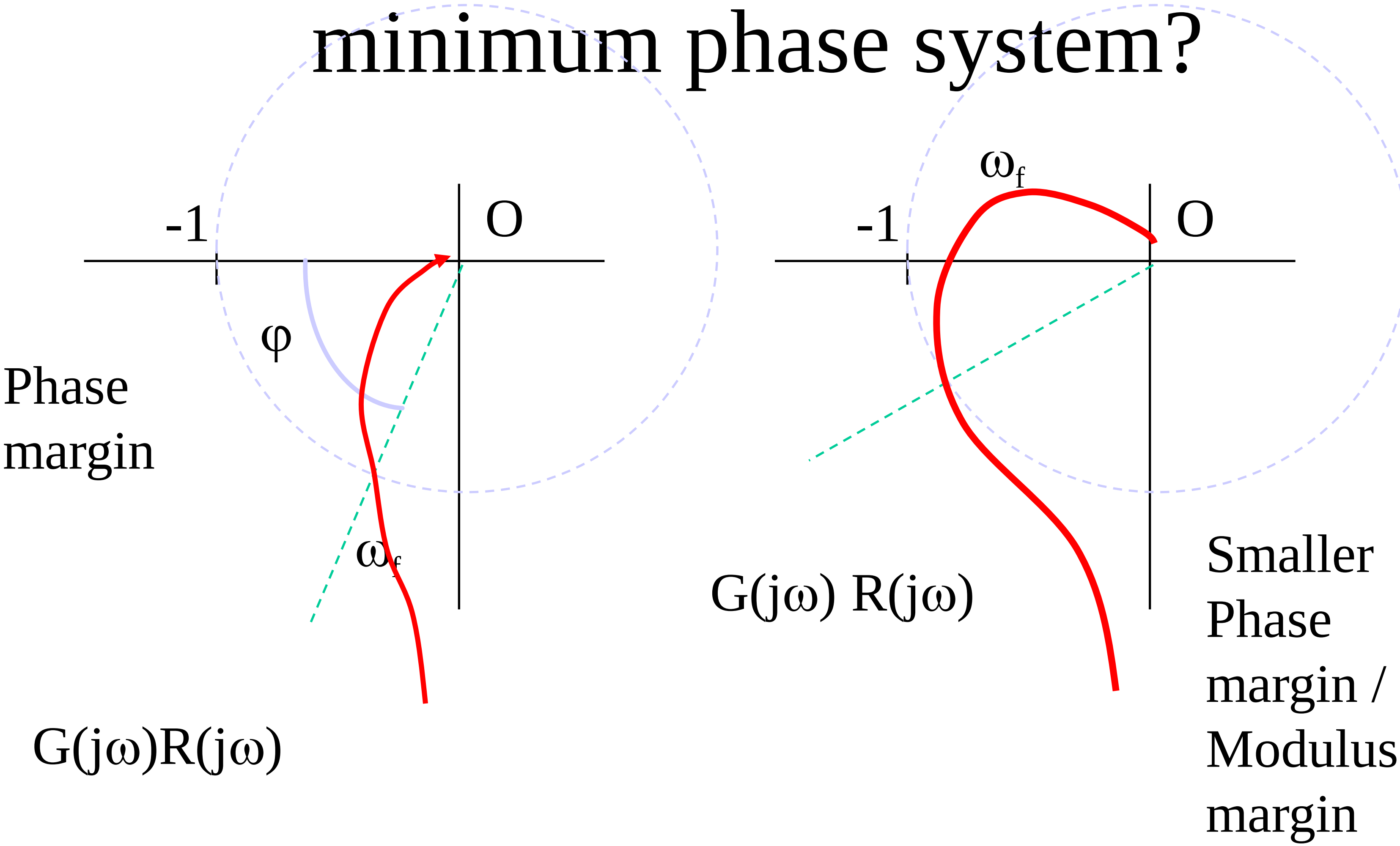
$$\arg(G(j\omega)) = \arg(K) + \arg(e^{-j\omega d}) + \arg(cj\omega - 1) + \ldots + \arg(1/j\omega) + \arg(1/(\tau j\omega + 1)) + \ldots$$



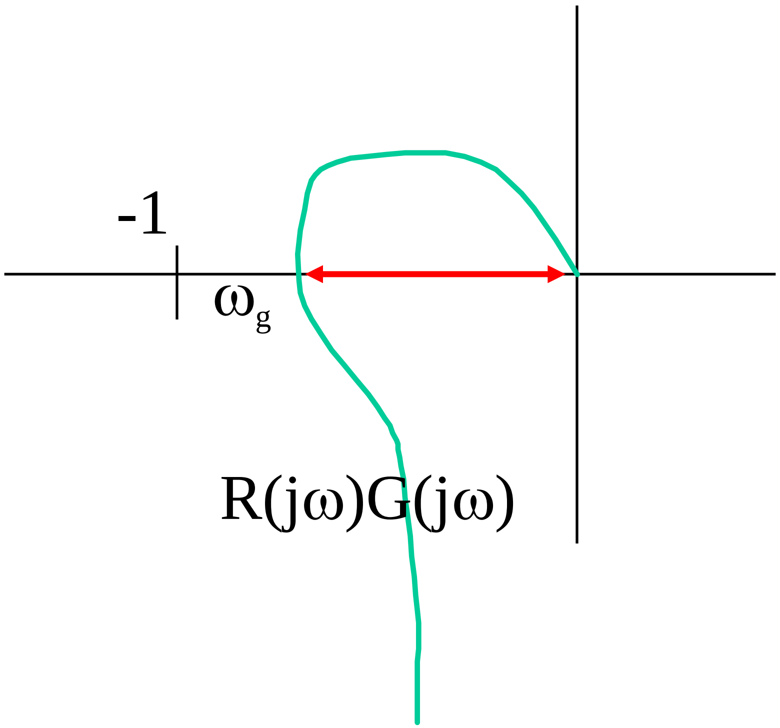
The zero increases the phase shift



Why is it difficult to control a non-minimum phase system?



Why should we choose the highest process gain for design when the gain of the process changes?



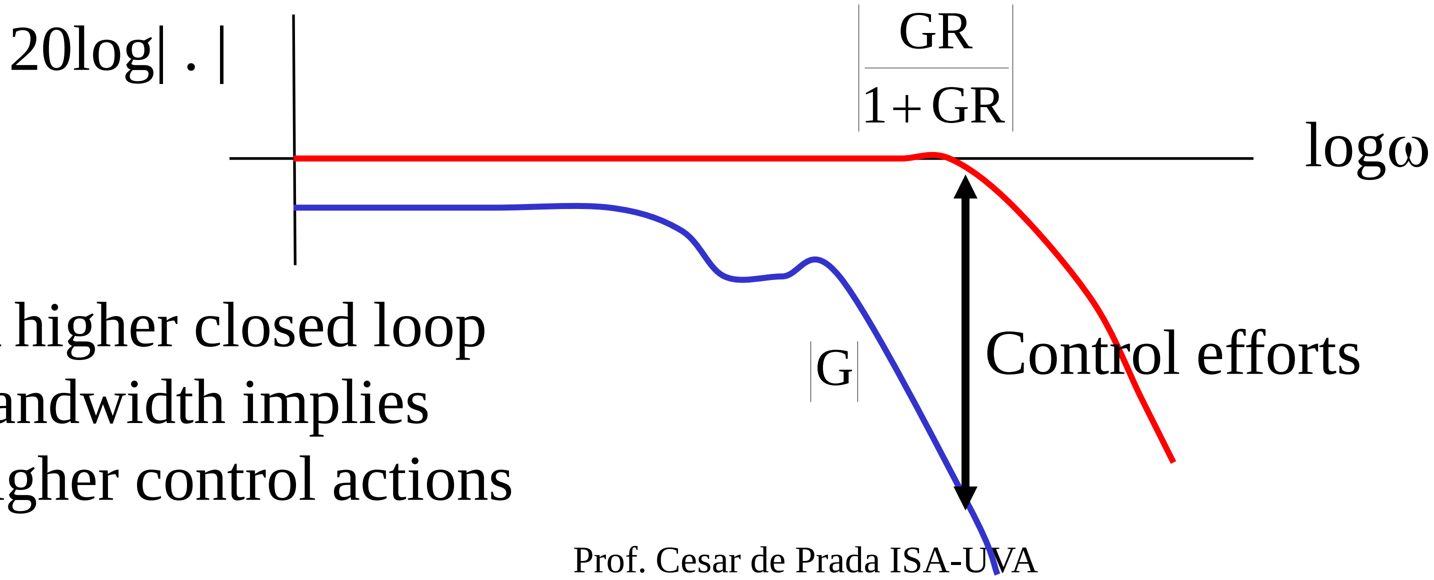
When the process gain changes, it will take a lower value and the gain margin will be increased (safest side).

If we choose the smaller process gain for designing a controller with the same gain margin, then, if the process gain increases, the gain margin will decrease.

Control efforts

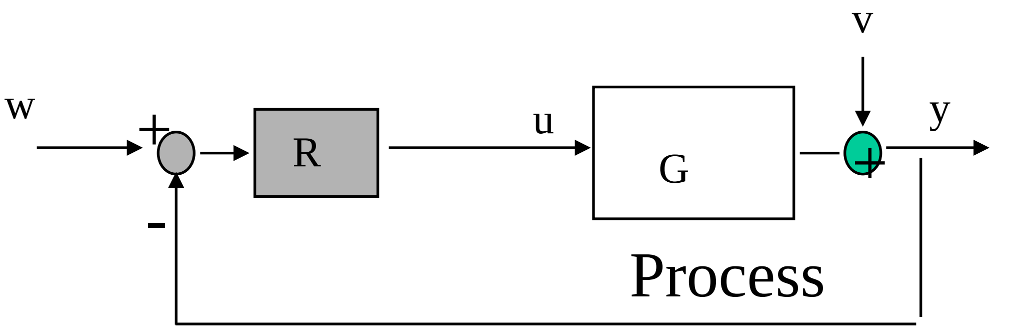
$$S_{wy} = \frac{GR}{1+GR} = G\frac{R}{1+GR} = G\,S_{wu}$$

$$20\log\left|\frac{GR(j\omega)}{1+GR(j\omega)}\right| - 20\log\left|G(j\omega)\right| = 20\log\left|\frac{R(j\omega)}{1+GR(j\omega)}\right|$$



A higher closed loop bandwidth implies higher control actions

Robustness / Sensibility

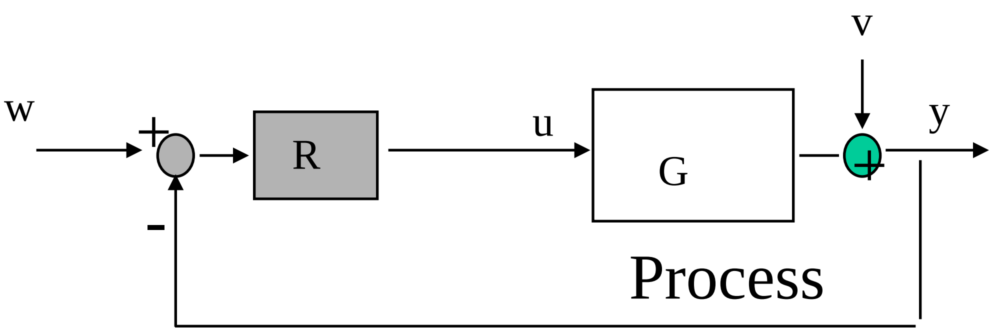


$$y = \frac{GR}{1+GR} w + \frac{1}{1+GR} v$$

How much the closed loop dynamics changes when the process transfer function changes?

Sensibility $\dfrac{\dfrac{\partial T}{T}}{\dfrac{\partial G}{G}} = \dfrac{G}{T}\dfrac{\partial T}{\partial G}$ $\qquad T = \dfrac{GR}{1+GR}$

Robustness / Sensibility



$$y = \frac{GR}{1+GR}w + \frac{1}{1+GR}v$$
$$= Tw + Sv$$

$$\frac{G}{T}\frac{\partial}{\partial G}\left[\frac{GR}{1+GR}\right] = \frac{G(1+GR)}{GR}\frac{(1+GR)R - GRR}{(1+GR)^2} = \frac{1+GR}{R}\frac{R}{(1+GR)^2} = \frac{1}{(1+GR)} = S_{vy}$$

$$\frac{G}{S}\frac{\partial}{\partial G}\left[\frac{1}{1+GR}\right] = \frac{G(1+GR)}{1}\frac{(-R)}{(1+GR)^2} = \frac{-GR}{(1+GR)} = -T$$

Sensibility function S_{vy} = sensibility against changes in G

Térmicos