Time response of dynamical systems

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Outline

- Introduction: Use of models
- Time response of first order systems
- Time response of second order systems
- Introduction to systems identification
- Time response of higher order systems
- Introduction to stability

Model based….

Analysis

The characteristics of the system response are deduced from the model

Design

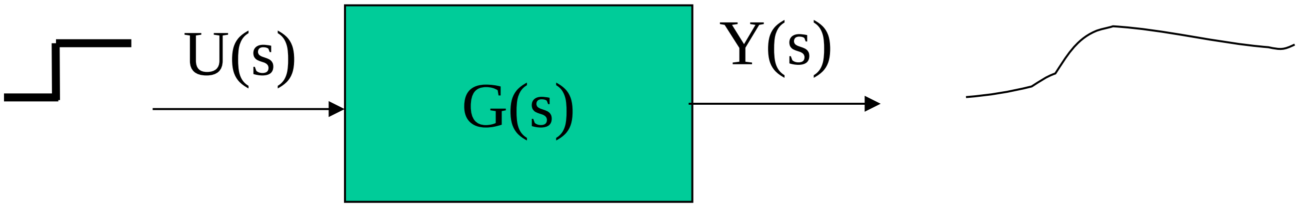
The process or the controller are designed using the model and the specifications

Control

The model is used explicitly in the controller for the control signal computation

Time response

Normalized signals

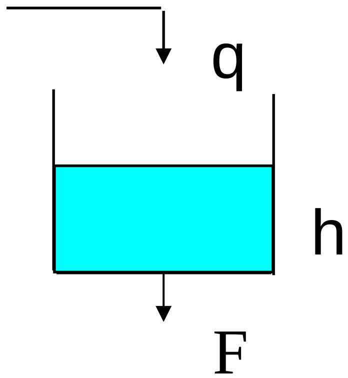


1 time → s transform → time

2 Deduce time response characteristics directly from the transfer function G(s)

3 Identification: Infer the model G(s) directly from experimental data

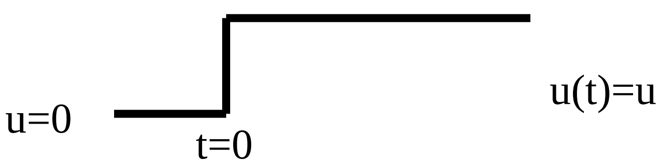
First order systems



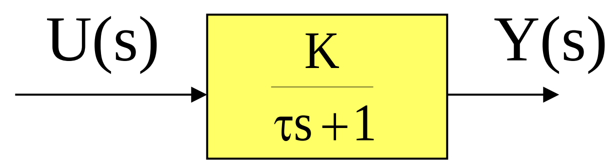
$$\tau \frac{d\Delta h}{dt} + \Delta h = K\Delta q$$

$$\tau = \frac{A2\sqrt{h_0}}{k} \qquad K = \frac{2\sqrt{h_0}}{k}$$

$$\tau \frac{dy(t)}{dt} + y(t) = Ku(t)$$



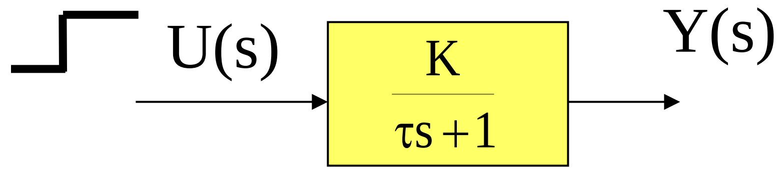
Transfer function:



Time response to a step jump in u starting from equilibrium

Step response

$$\tau \frac{dy(t)}{dt} + y(t) = Ku(t)$$



Partial fraction expansion

$$Y(s) = \frac{K}{(\tau s+1)}\frac{u}{s} = \frac{K/\tau}{(s+1/\tau)}\frac{u}{s} = \frac{\alpha}{s} + \frac{\beta}{s+1/\tau} = \frac{\alpha(s+1/\tau)}{s(s+1/\tau)} + \frac{\beta s}{s(s+1/\tau)}$$

$$\text{for } s = 0 \quad \Rightarrow Ku/\tau = \alpha/\tau; \qquad \alpha = Ku$$

$$\text{for } s = -1/\tau \quad \Rightarrow Ku/\tau = -\beta/\tau; \qquad \beta = -Ku$$

$$Y(s) = Ku\left(\frac{1}{s} - \frac{1}{s+1/\tau}\right); \qquad y(t) = L^{-1}[Y(s)] = Ku\left(L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{1}{s+1/\tau}\right]\right)$$

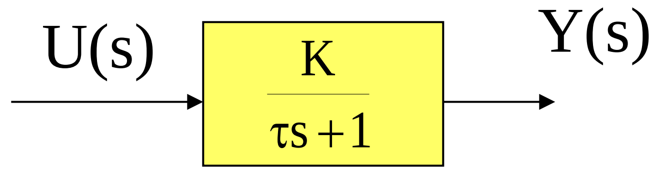
$$y(t) = Ku(1 - e^{-\frac{t}{\tau}})$$

Test:

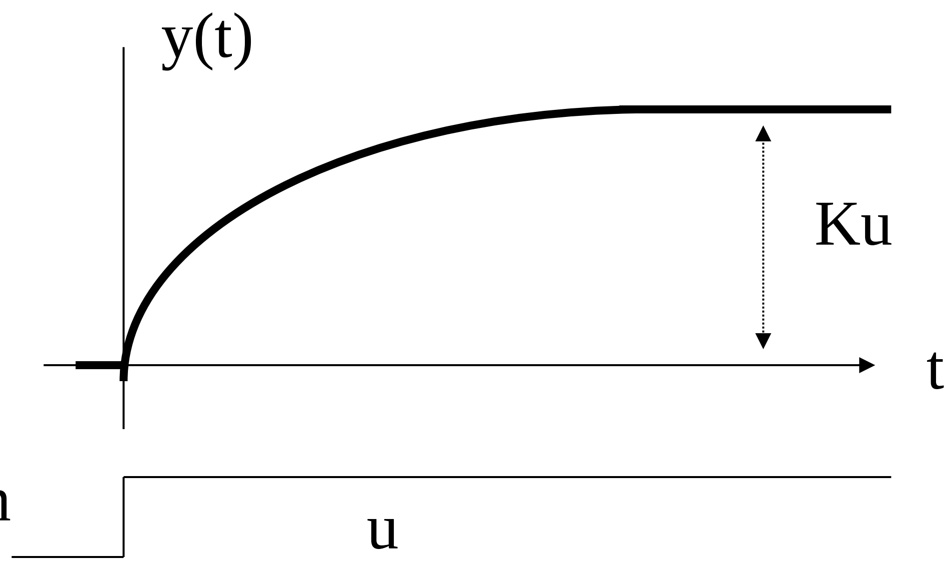
$$\tau\left[Ku\frac{e^{-\frac{t}{\tau}}}{\tau}\right] + Ku(1 - e^{-\frac{t}{\tau}}) = Ku$$

Step response

$$\tau \frac{dy(t)}{dt} + y(t) = Ku(t)$$



$$y(t) = Ku(1 - e^{-t/\tau})$$

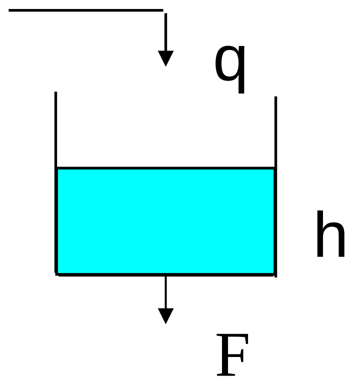


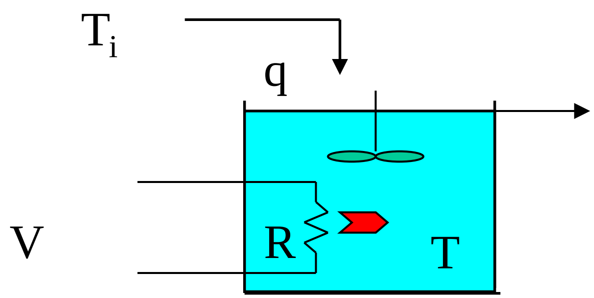
$\tau > 0$ time constant

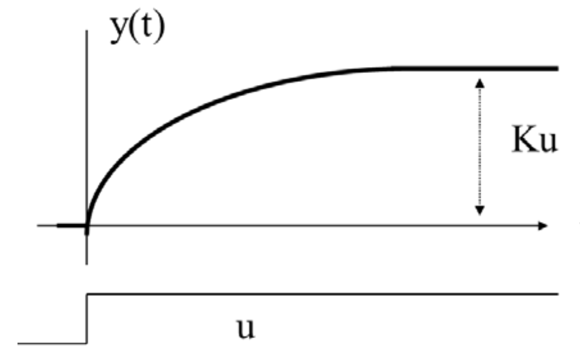
Time response stable, without delay nor change in concavity, and overdamped

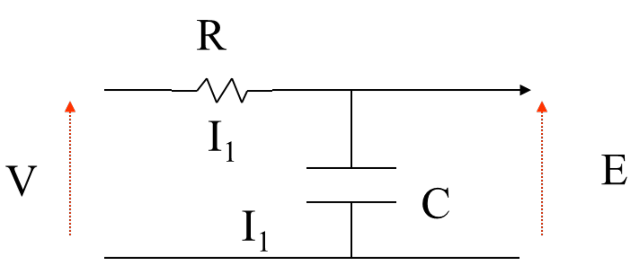
Gain = K = Ku/u

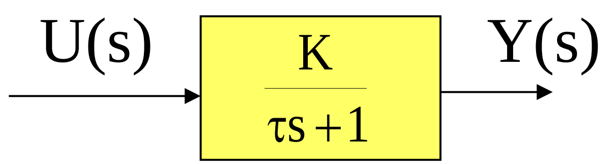
All First order systems responds in the same way

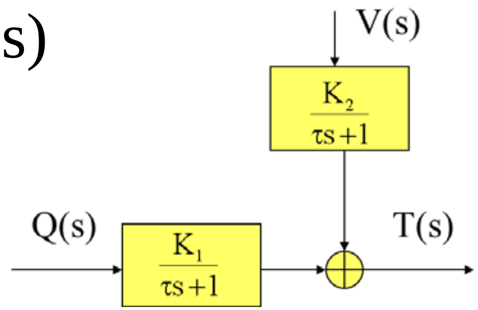






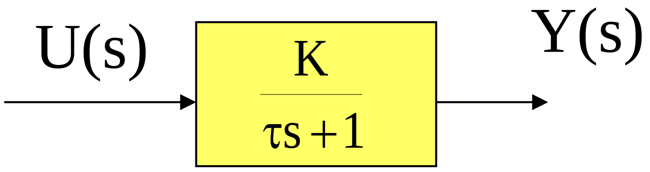






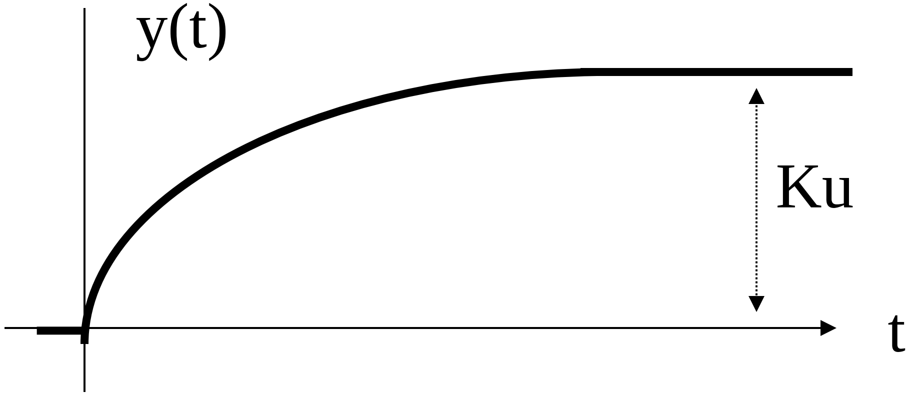
Interpretation in s

$$y(t) = Ku(1 - e^{-t/\tau})$$

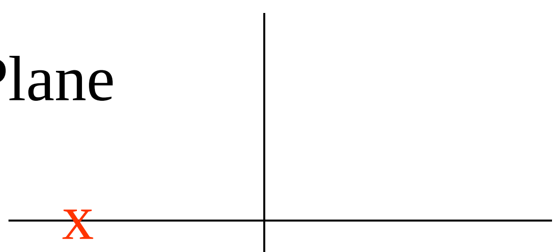


$\tau s+1=0$

pole = $-1/\tau$



s Plane

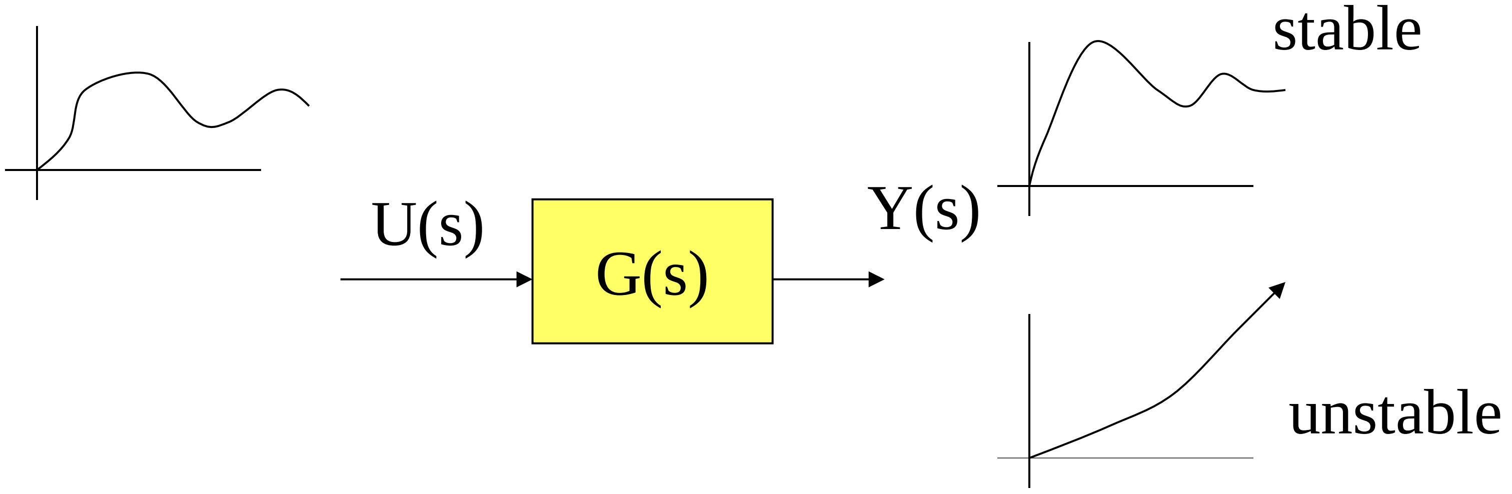


Pole located on the real axis, in the left hand side of the s plane

If $\tau > 0$ Time response stable, without change in convexity and overdamped

Input-Output stability (BIBO)

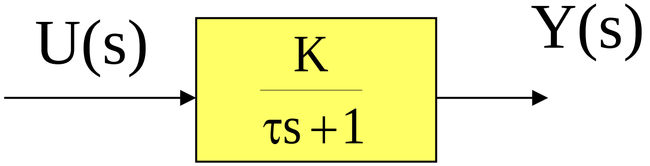
Bounded Input-Bounded Output



A system is input-output stable if its time response is bounded when the input is bounded too.

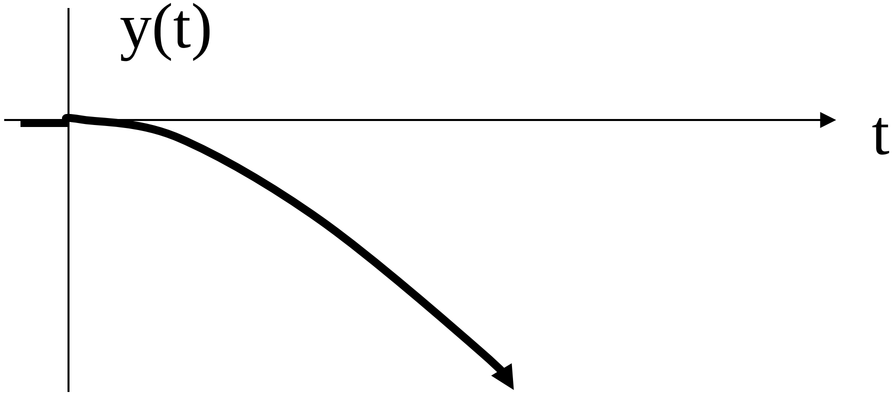
Interpretation en s (τ<0)

$$y(t) = Ku(1 - e^{-t/\tau})$$

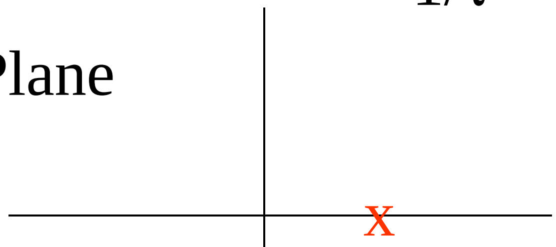


$\tau s + 1 = 0$

positive pole $= -1/\tau$



s Plane

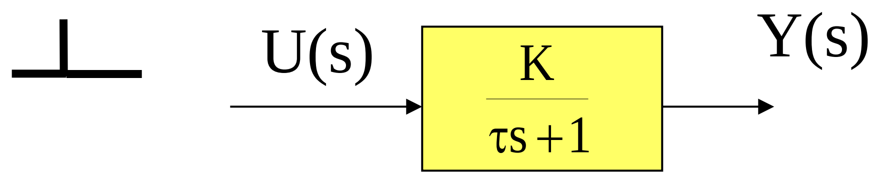


Pole located in the right half s plane

If $\tau < 0$
unstable time response

Using other inputs

Example: impulse



$$Y(s) = \frac{K}{(\tau s + 1)} u = \frac{K/\tau}{(s + 1/\tau)} u$$

$$y(t) = L^{-1}[Y(s)] = \frac{Ku}{\tau} L^{-1}\left[\frac{1}{s + 1/\tau}\right]$$

$$y(t) = \frac{Ku}{\tau} e^{-\frac{t}{\tau}}$$

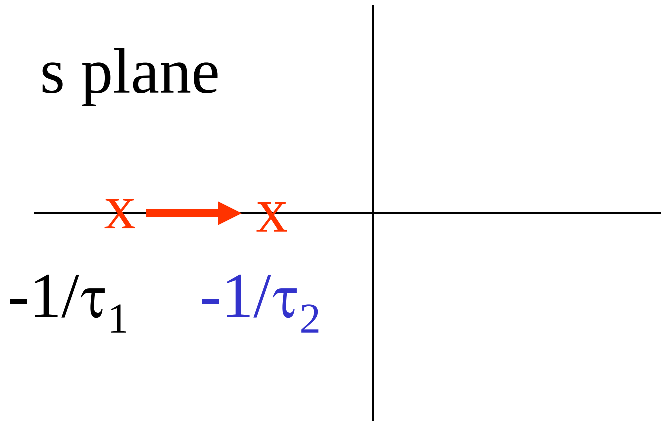
Stability is determined by the pole location, not for the type of input

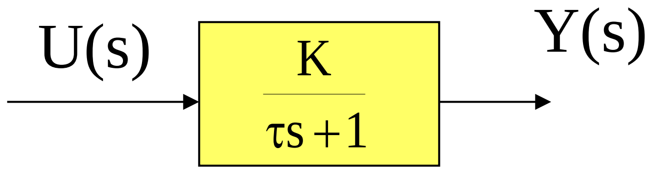
Partial fraction expansion

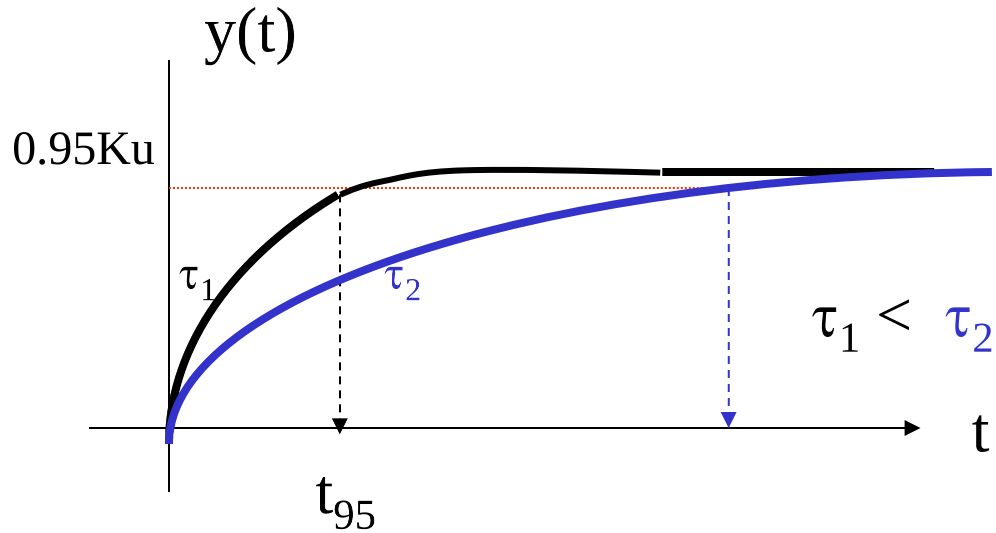
$$Y(s) = \frac{K}{(\tau s + 1)} U(s) = \frac{a}{(s + 1/\tau)} + \ldots.$$

$$y(t) = L^{-1}[Y(s)] = L^{-1}\left[\frac{a}{s + 1/\tau}\right] + L^{-1}[\ldots] = a e^{-\frac{t}{\tau}} + \ldots$$

Settling time







$$y(t_{95}) = 0.95Ku = Ku(1 - e^{-\frac{t_{95}}{\tau}})$$

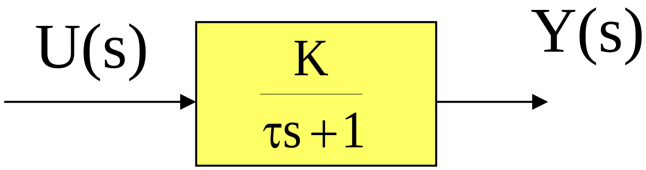
$$t_{95} = 3\tau$$

$$y(t) = Ku(1 - e^{\frac{-t}{\tau}})$$

Time constant

$$y(t) = Ku(1 - e^{\frac{-t}{\tau}})$$

$$y(\tau) = Ku(1 - e^{-1}) = 0.632Ku$$

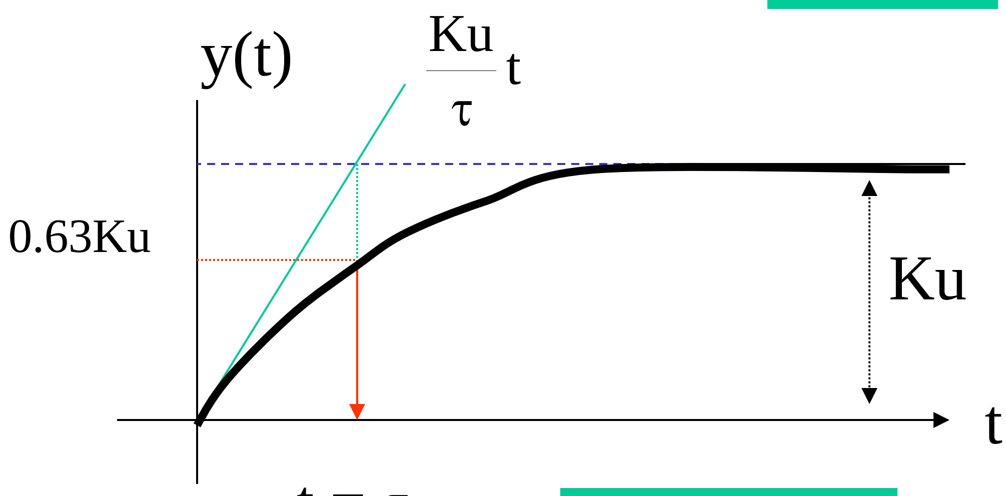


resp

Derivative at the origin

$$\frac{d\,y(t)}{dt} = \frac{Ku}{\tau}(e^{\frac{-t}{\tau}})$$

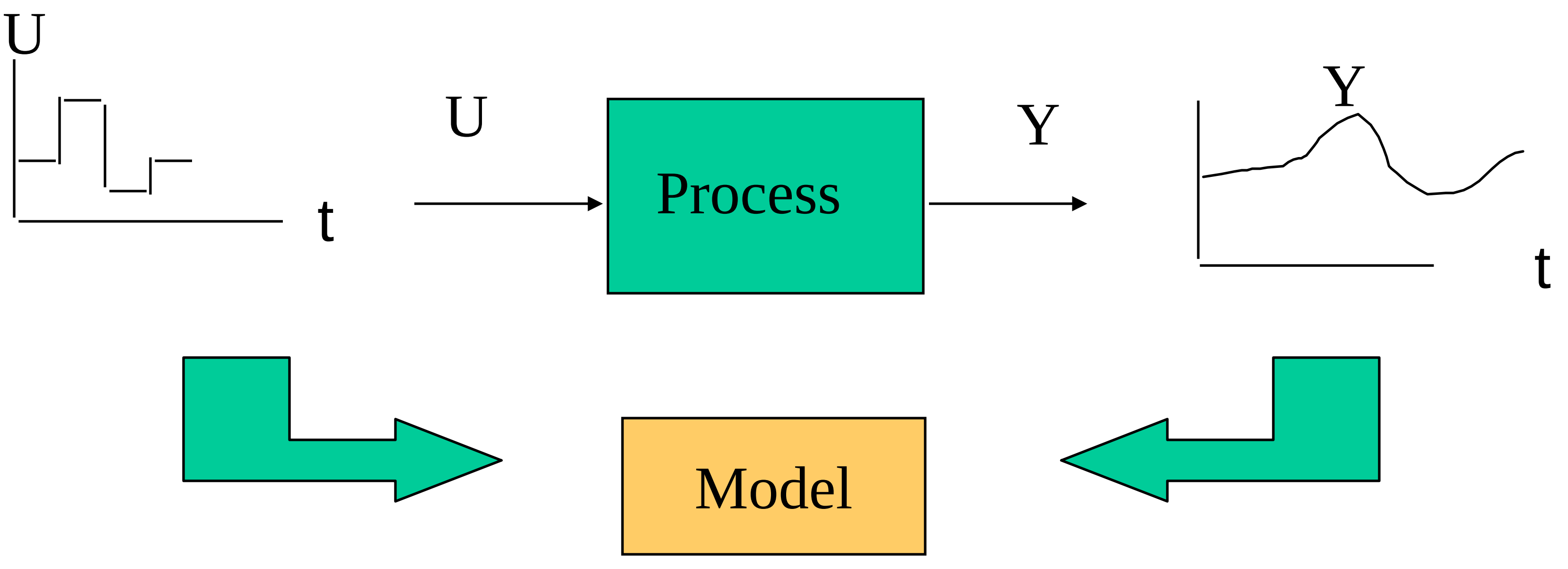
$$\left.\frac{d\,y(t)}{dt}\right|_{t=0} = \frac{Ku}{\tau}$$



SysQuake

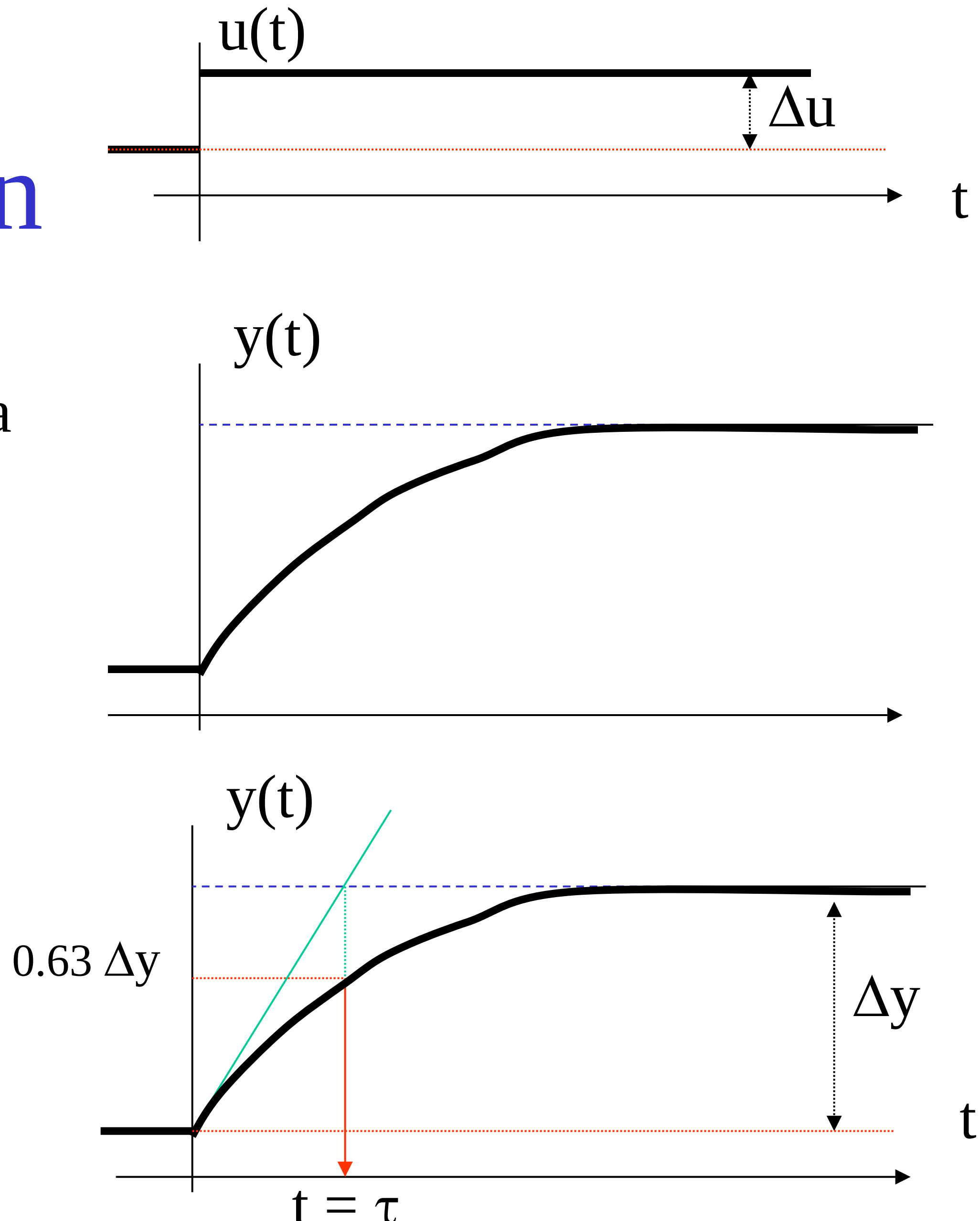
Identification

The model is obtained from input-output experimental data



Identification

If the time response to a input step Δu starting from an equilibrium point is like the one in the figure ⇒ first order system



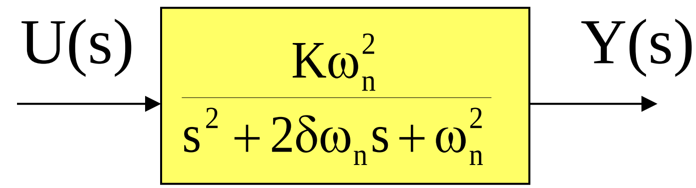
Parameter estimation:

$K = \Delta y / \Delta u$

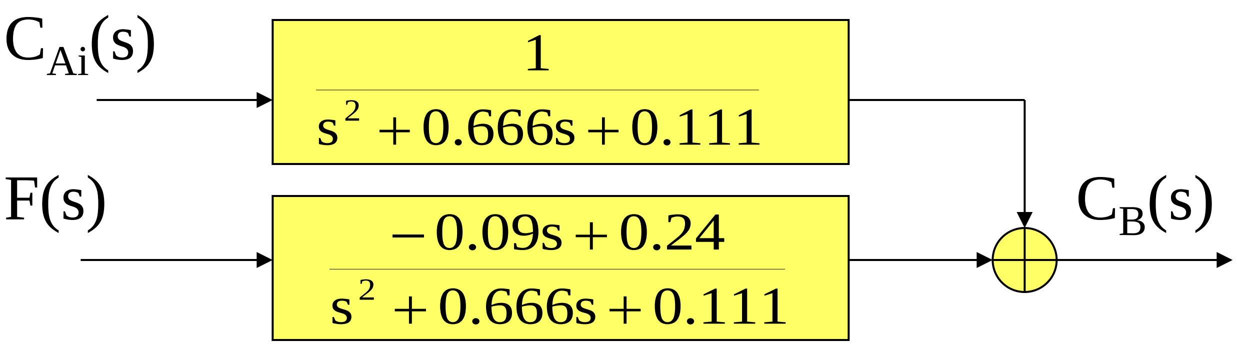
τ Two methods

Second order systems

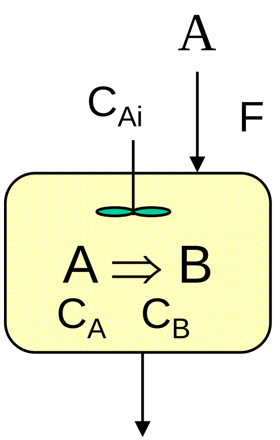
$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{13} & a_{14} \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} u$$



$$y = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

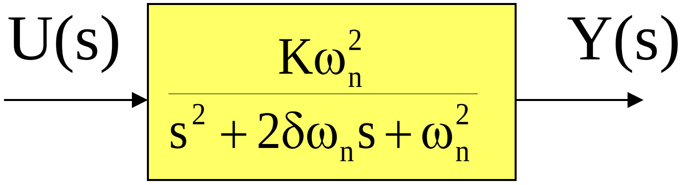


Isothermal reactor

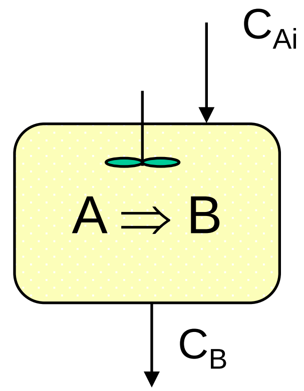


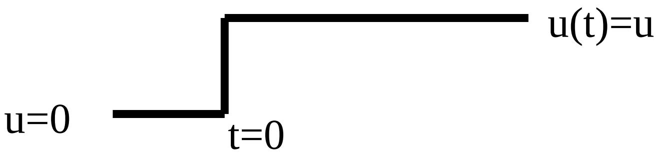
Second order systems

$$\frac{d^2y(t)}{dt^2} + 2\delta\omega_n \frac{dy(t)}{dt} + \omega_n^2 y(t) = K\omega_n^2 u(t)$$



$$\frac{K\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}$$





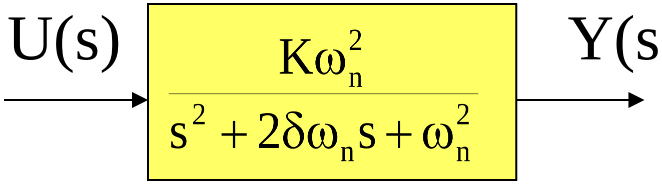
Step response in u(t)

K gain

δ Damping ratio

ω_n (undamped) natural frequency

Second order systems



Poles:

$$s^2+2\delta\omega_n s+\omega_n^2=0$$

$$s=\frac{-2\delta\omega_n \pm \sqrt{4\delta^2\omega_n^2-4\omega_n^2}}{2}=-\delta\omega_n \pm \omega_n\sqrt{\delta^2-1}$$

$$\text{si} \quad \omega_n>0, \quad \delta>0$$

$$\text{if} \quad \delta \geq 1 \qquad 2 \text{ negative real roots}$$

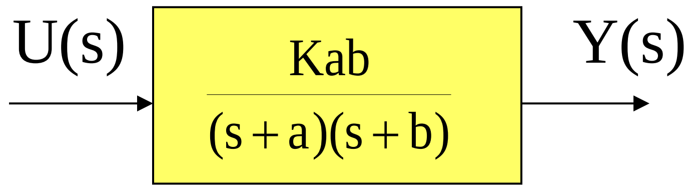
$$\text{if} \quad \delta < 1 \qquad 2 \text{ complex conjugate roots}$$

$$-\delta\omega_n \pm j\omega_n\sqrt{1-\delta^2}$$

Step response, δ >1

$$a = \delta\omega_n - \omega_n\sqrt{\delta^2 - 1}$$

$$b = \delta\omega_n + \omega_n\sqrt{\delta^2 - 1}$$



$$Y(s) = \frac{Kab}{(s+a)(s+b)}\frac{u}{s} = \frac{\alpha}{s} + \frac{\beta}{s+a} + \frac{\gamma}{s+b} =$$

$$= \frac{\alpha(s+a)(s+b)}{s(s+a)(s+b)} + \frac{\beta s(s+b)}{s(s+a)(s+b)} + \frac{\gamma s(s+a)}{s(s+a)(s+b)}$$

$$\text{for } s = 0 \quad \Rightarrow Kabu = \alpha ab \qquad \alpha = Ku$$

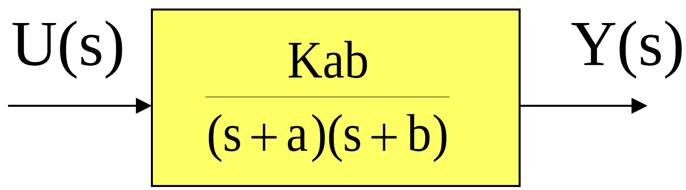
$$\text{for } s = -a \quad \Rightarrow Kabu = \beta(-a)(-a+b) \qquad \beta = Kub/(a-b) = Ku\frac{-\delta - \sqrt{\delta^2-1}}{2\sqrt{\delta^2-1}}$$

$$\text{for } s = \text{-}b \quad \Rightarrow Kabu = \gamma(\text{-}b)(\text{-}b+a) \qquad \gamma = -Kua/(a\text{-}b) = Ku\frac{-\delta + \sqrt{\delta^2-1}}{2\sqrt{\delta^2-1}}$$

Step response, $\delta > 1$

$$a = \delta\omega_n - \omega_n\sqrt{\delta^2 - 1}$$

$$b = \delta\omega_n + \omega_n\sqrt{\delta^2 - 1}$$



$$\frac{K}{(\frac{1}{a}s+1)(\frac{1}{b}s+1)}$$

2 time constants 1/a, 1/b

$$Y(s) = (\frac{\alpha}{s} + \frac{\beta}{s+a} + \frac{\gamma}{s+b});$$

$$y(t) = L^{-1}[Y(s)] = L^{-1}\left[\frac{\alpha}{s}\right] + L^{-1}\left[\frac{\beta}{s+a}\right] + L^{-1}\left[\frac{\gamma}{s+b}\right]$$

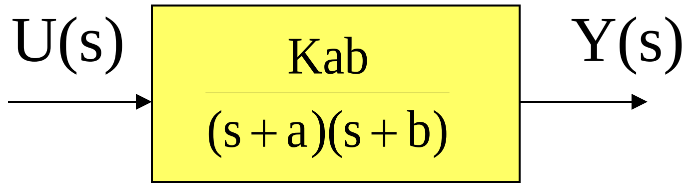
$$y(t) = \alpha + \beta e^{-at} + \gamma e^{-bt} = Ku(1 + \frac{-\delta - \sqrt{\delta^2 - 1}}{2\sqrt{\delta^2 - 1}}e^{-at} - \frac{-\delta + \sqrt{\delta^2 - 1}}{2\sqrt{\delta^2 - 1}}e^{-bt})$$

$y(0) = 0$ $\qquad$ $y(\infty) = Ku$ $\qquad$ monotonously increasing function

Step response, $\delta > 1$

$$a = \delta\omega_n - \omega_n\sqrt{\delta^2 - 1}$$

$$b = \delta\omega_n + \omega_n\sqrt{\delta^2 - 1}$$

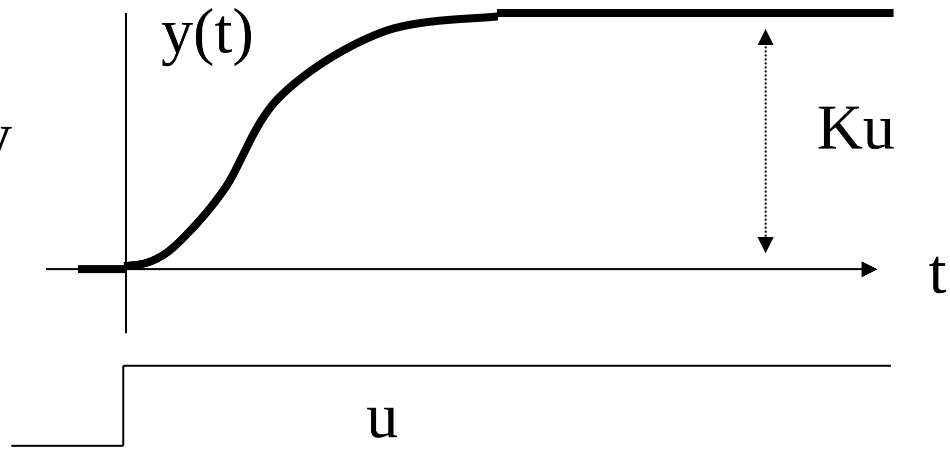


$$\frac{K}{(\frac{1}{a}s+1)(\frac{1}{b}s+1)}$$

$$y(t) = \alpha + \beta e^{-at} + \gamma e^{-bt} = Ku\left(1 + \frac{-\delta - \sqrt{\delta^2-1}}{2\sqrt{\delta^2-1}}e^{-at} - \frac{-\delta + \sqrt{\delta^2-1}}{2\sqrt{\delta^2-1}}e^{-bt}\right)$$

Time response stable, without delay, with concavity change and overdamped

Gain = K = Ku/u

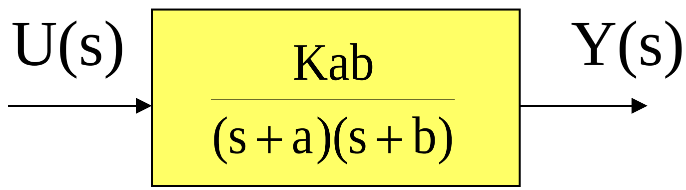


Interpretation in s

respx

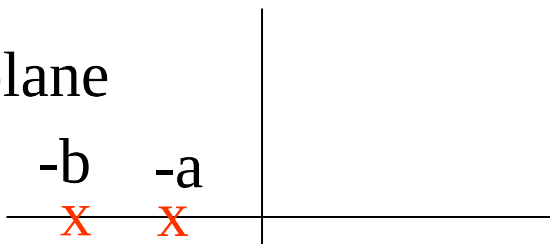
Dominant poles
Concavity

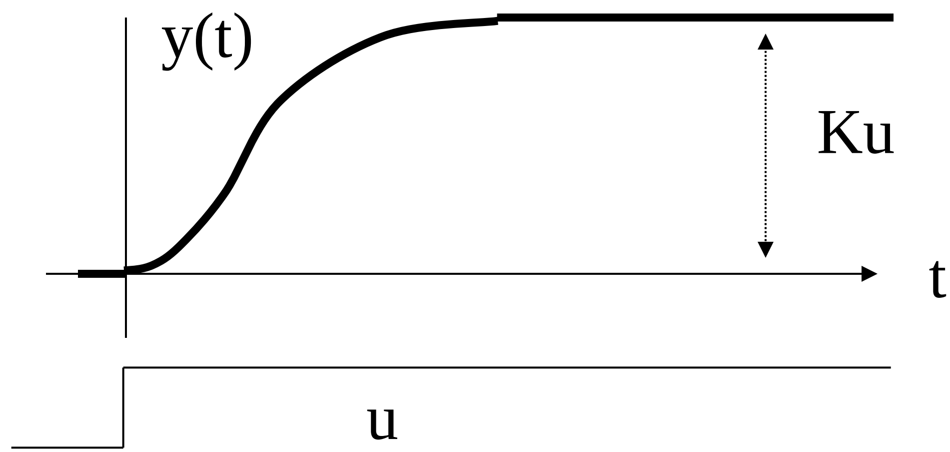
SysQuake



The right most pole dominates the transient decrease

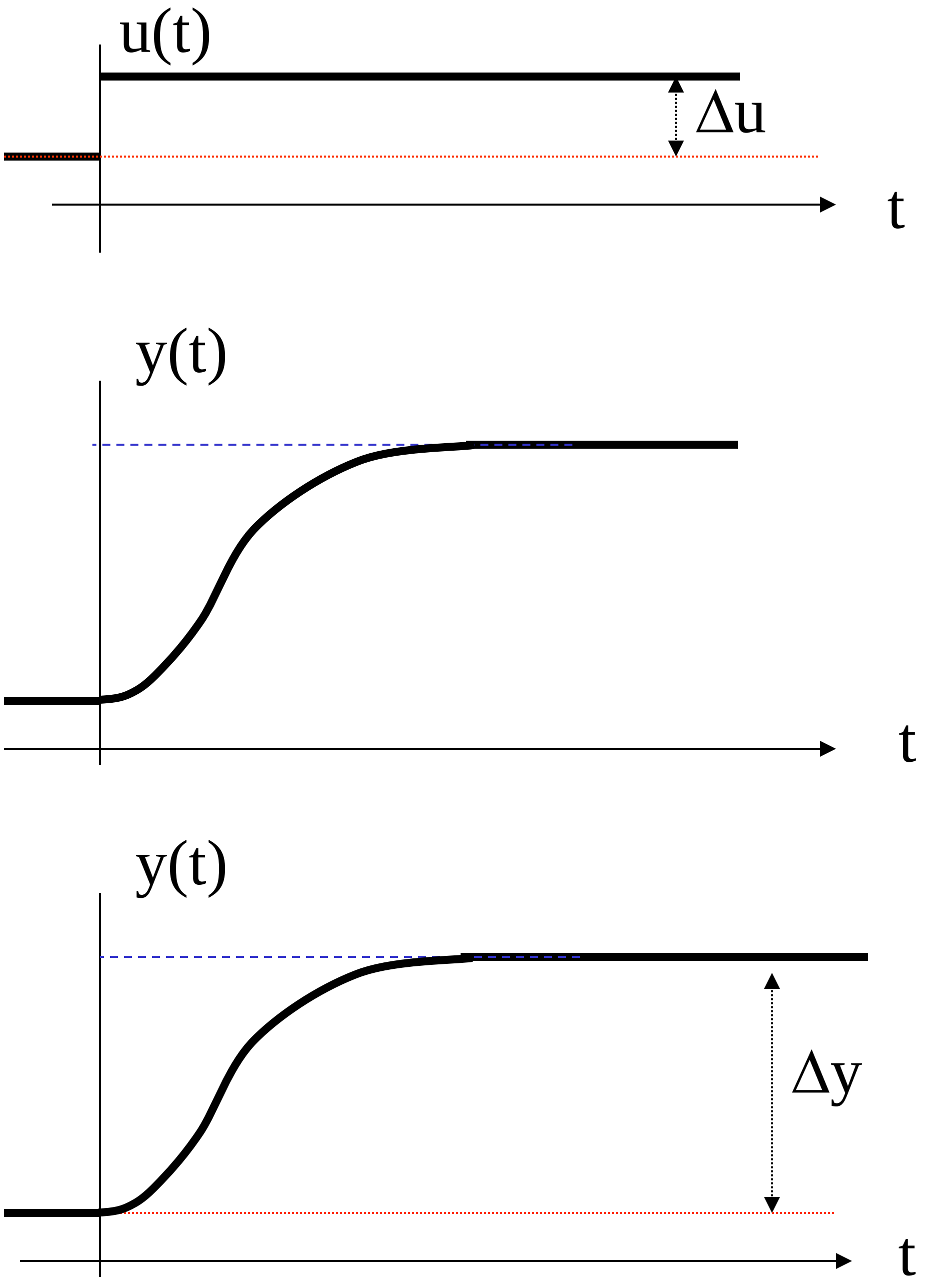
$$y(t) = \alpha + \beta e^{-at} + \gamma e^{-bt}$$





Poles located on the real axis in the left hand side of the s plane

Identification



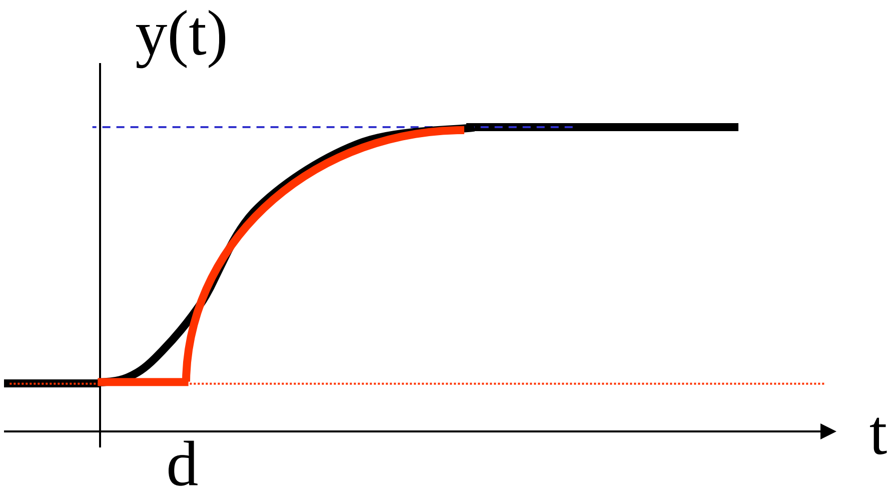
If the time response to a input step Δu starting from an equilibrium point is like the one in the figure ⇒ second order system with real poles

Parameter estimation:

$K = \Delta y / \Delta u$

Time constants difficult to estimate

Approximation

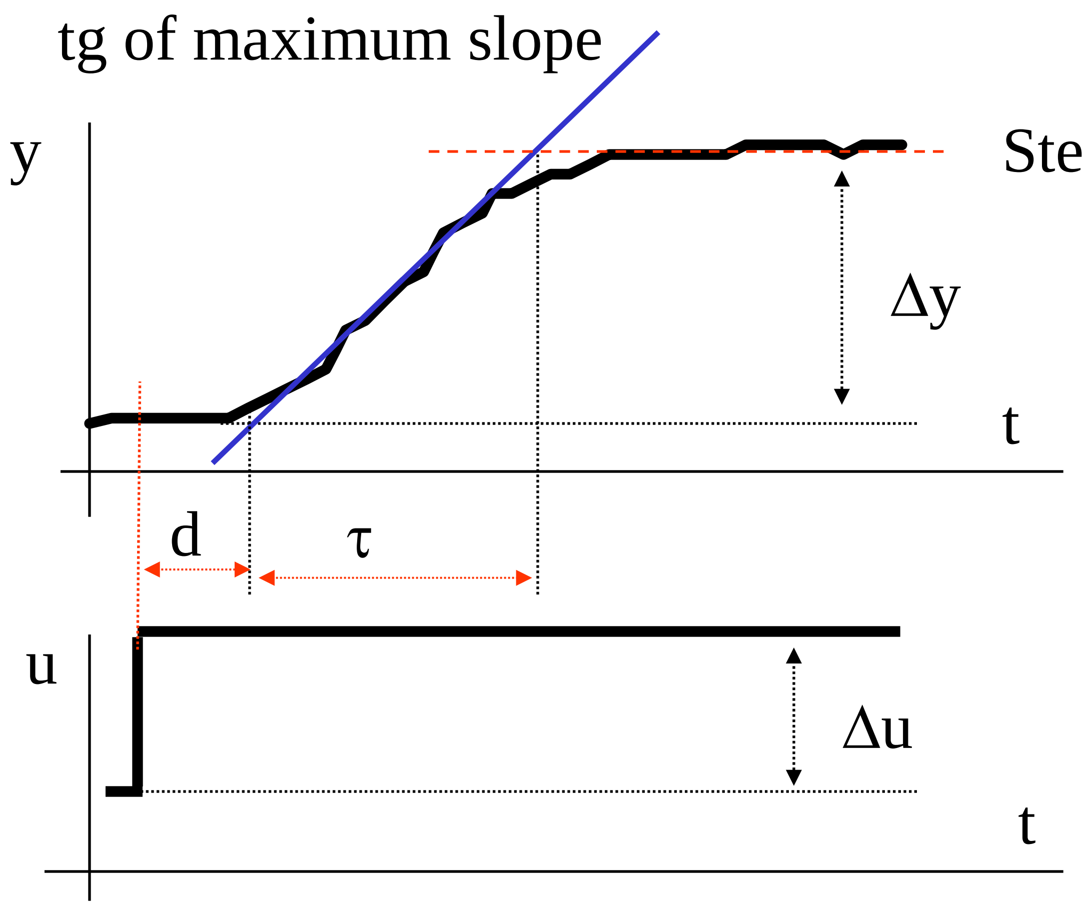


$$\frac{Kab}{(s+a)(s+b)}$$

$$\frac{Ke^{-ds}}{\tau s+1}$$

The time response of an overdamped second order system can be approximated by the one of a first order plus delay system

Identification using the step response



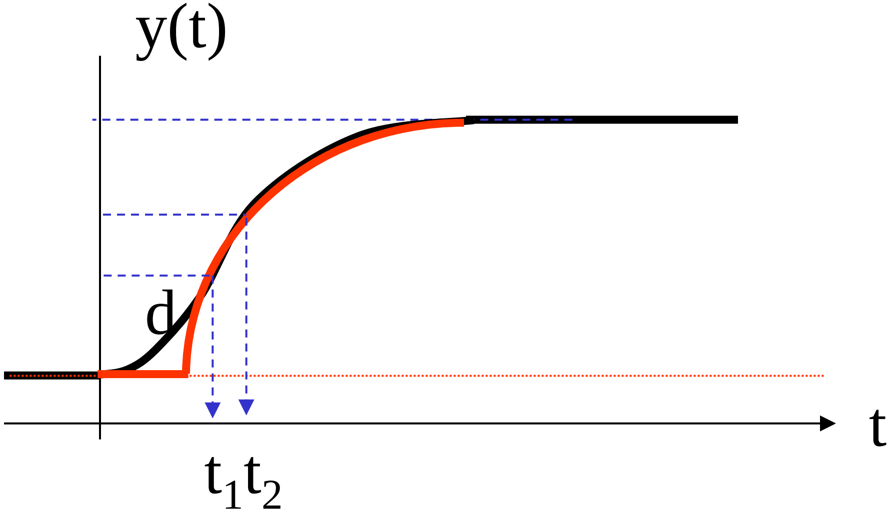
K= Δy/Δu

$$\frac{Ke^{-ds}}{\tau s + 1}$$

Identification of FOPD systems

$$\frac{Ke^{-ds}}{\tau s+1}$$

$$y(t) = Ku(1-e^{\frac{-t+d}{\tau}})$$



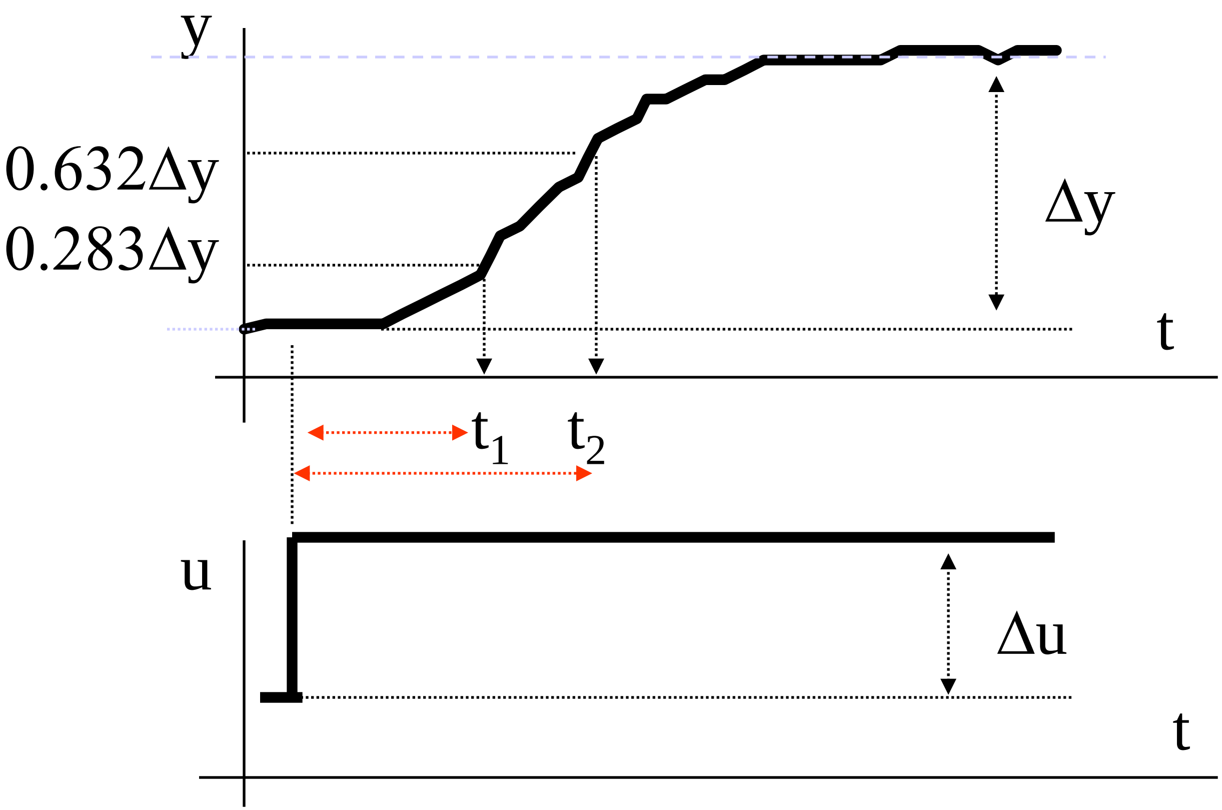
Computing the time response at time instants: $t_2 = d+\tau$ and $t_1 = d+\tau/3$:

$$y(t) = Ku(1-e^{-1}) = 0.632Ku$$

$$y(t) = Ku(1-e^{-1/3}) = 0.283Ku$$

Estimating $t_1 = d+\tau$ and $t_2 = d+\tau/3$ from the time response, one can compute d and τ

Identification using the step response



$\tau = 1.5\,(t_2 - t_1)$

$d = t_2 - \tau$

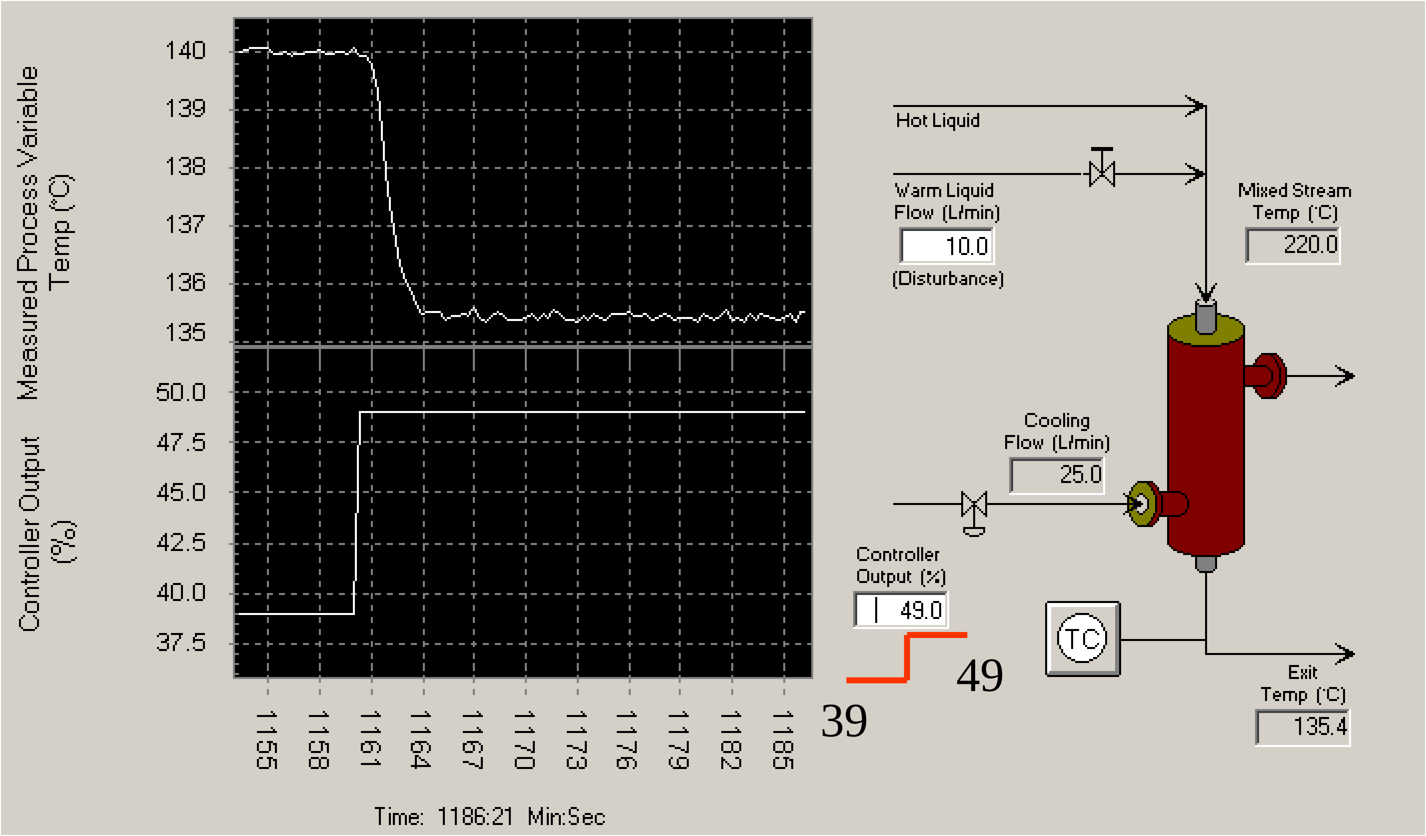
$K = \Delta y / \Delta u$

$$\frac{Ke^{-ds}}{\tau s + 1}$$

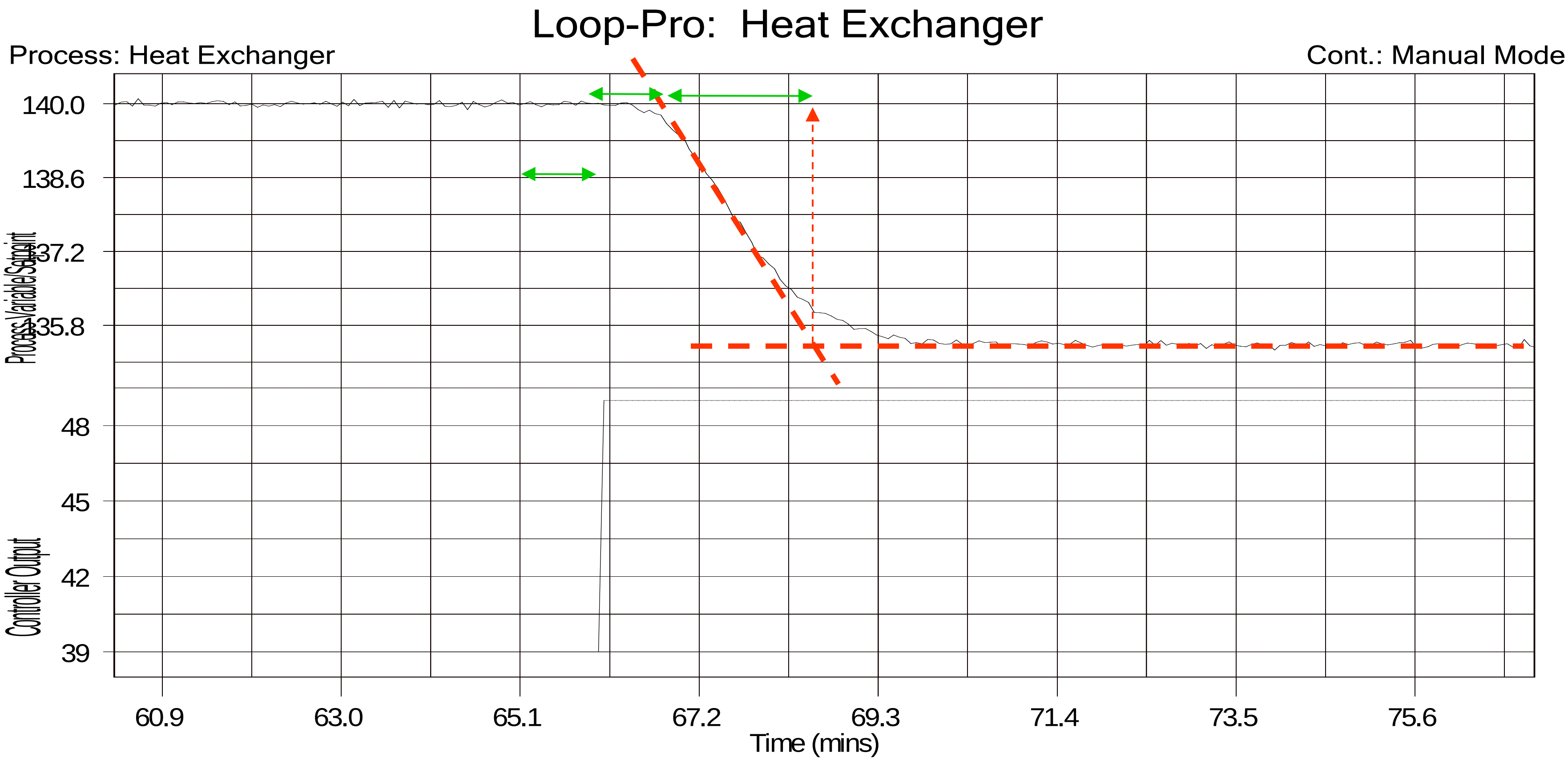
Problema03

Heat exchanger

Open loop test



Heat exchanger

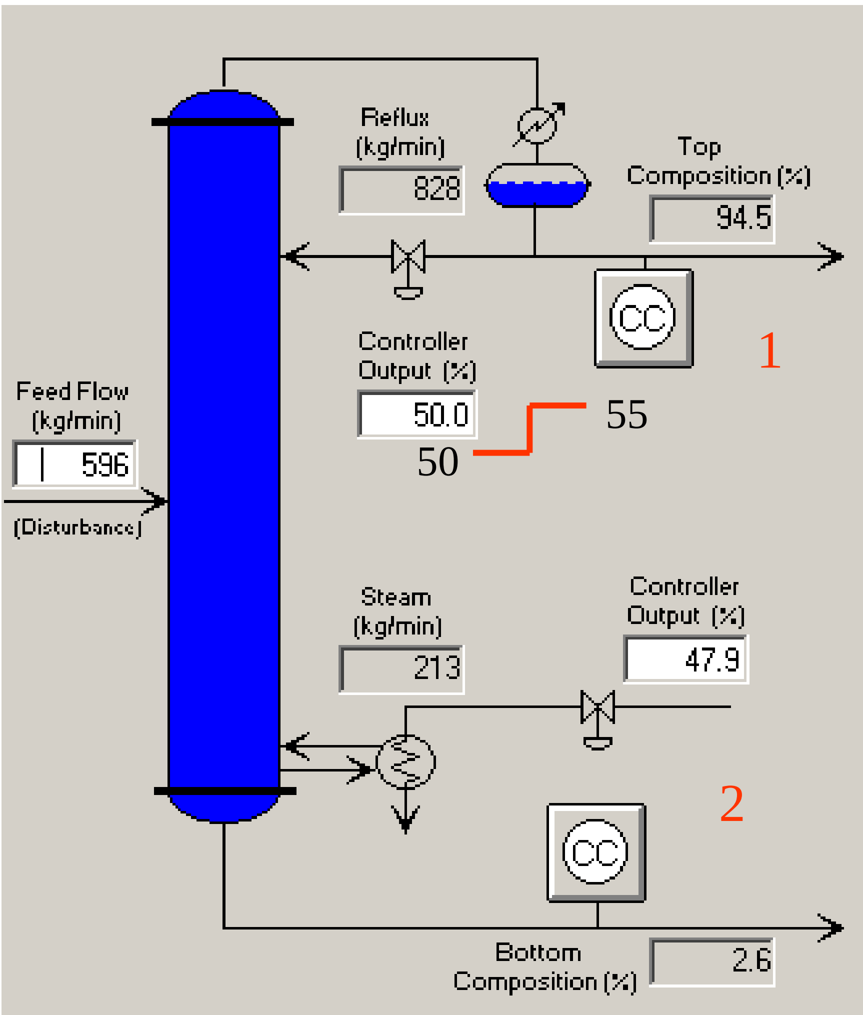


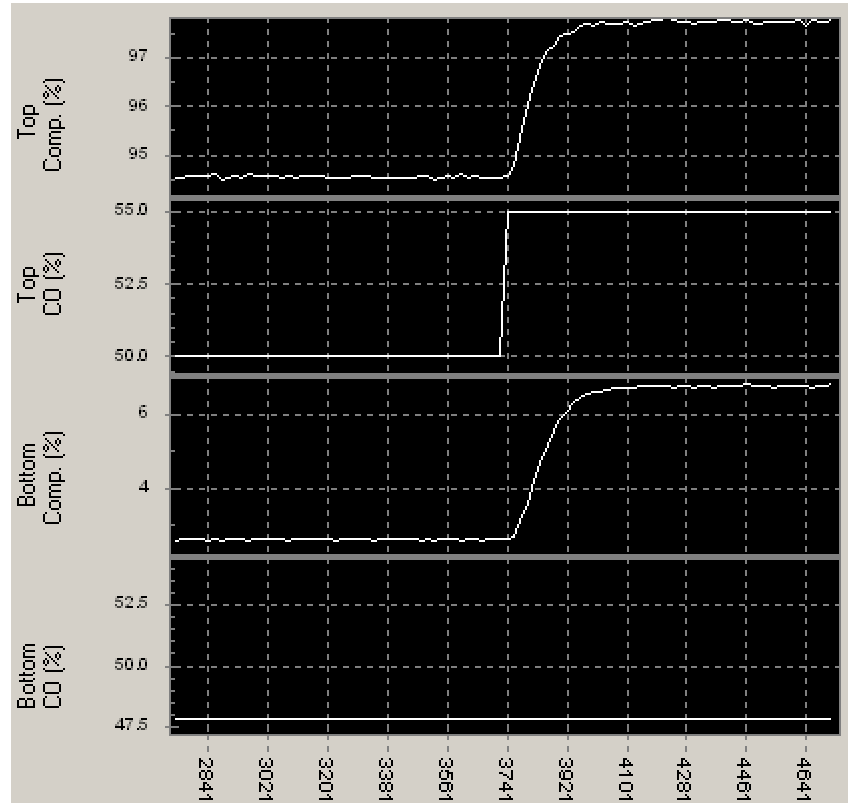
K = (135.4 -140) /10 = -0.46

D = 0.75   τ = 1.4

$$G(s) = \frac{-0.46e^{-0.75s}}{1.4s+1}$$

Multivariable Destilation column

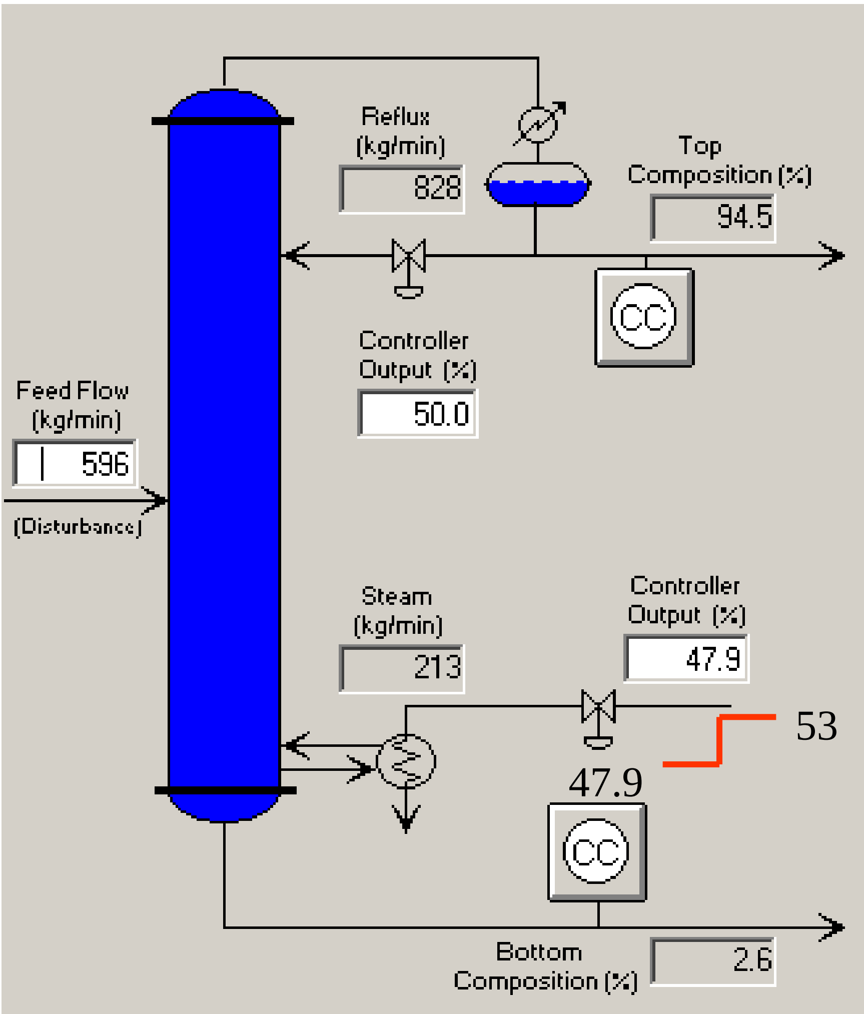


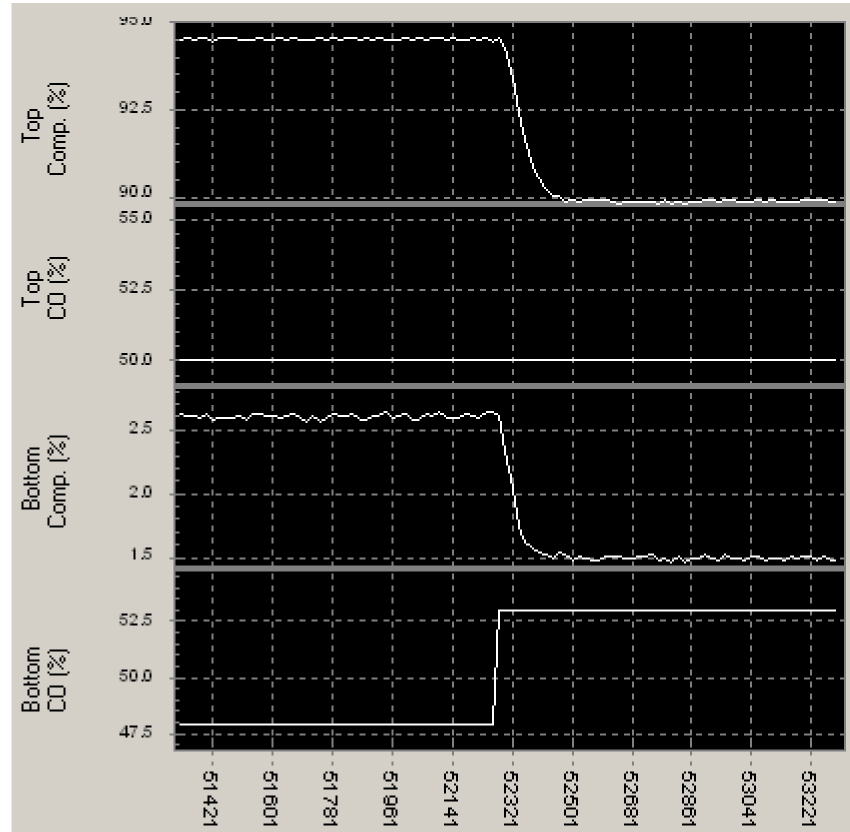


$$G_{11} = \frac{K_{11} e^{-d_{11} s}}{\tau_{11} s + 1} = \frac{0.648 e^{-21.7s}}{60s + 1}$$

$$G_{21} = \frac{K_{21} e^{-d_{21} s}}{\tau_{21} s + 1} = \frac{0.815 e^{-34.4s}}{84.7s + 1}$$

Other MV



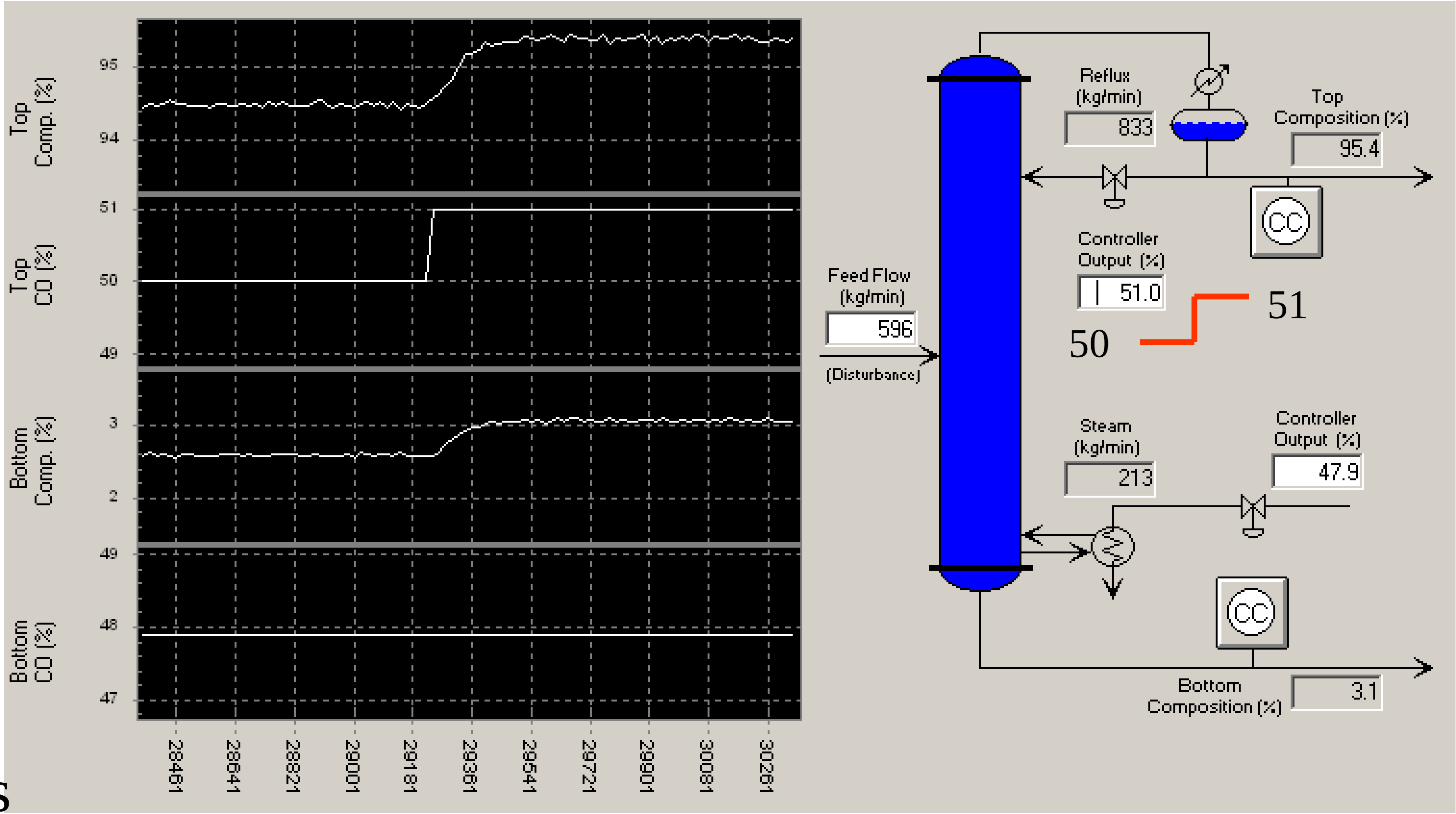


$$G_{12} = \frac{K_{12}e^{-d_{12}s}}{\tau_{12}s+1} = \frac{-0.894e^{-21.6s}}{54.3s+1}$$

$$G_{22} = \frac{K_{22}e^{-d_{22}s}}{\tau_{22}s+1} = \frac{-0.236e^{-6.61s}}{41.9s+1}$$

Repeated experiment with smaller changes

Change of 1%

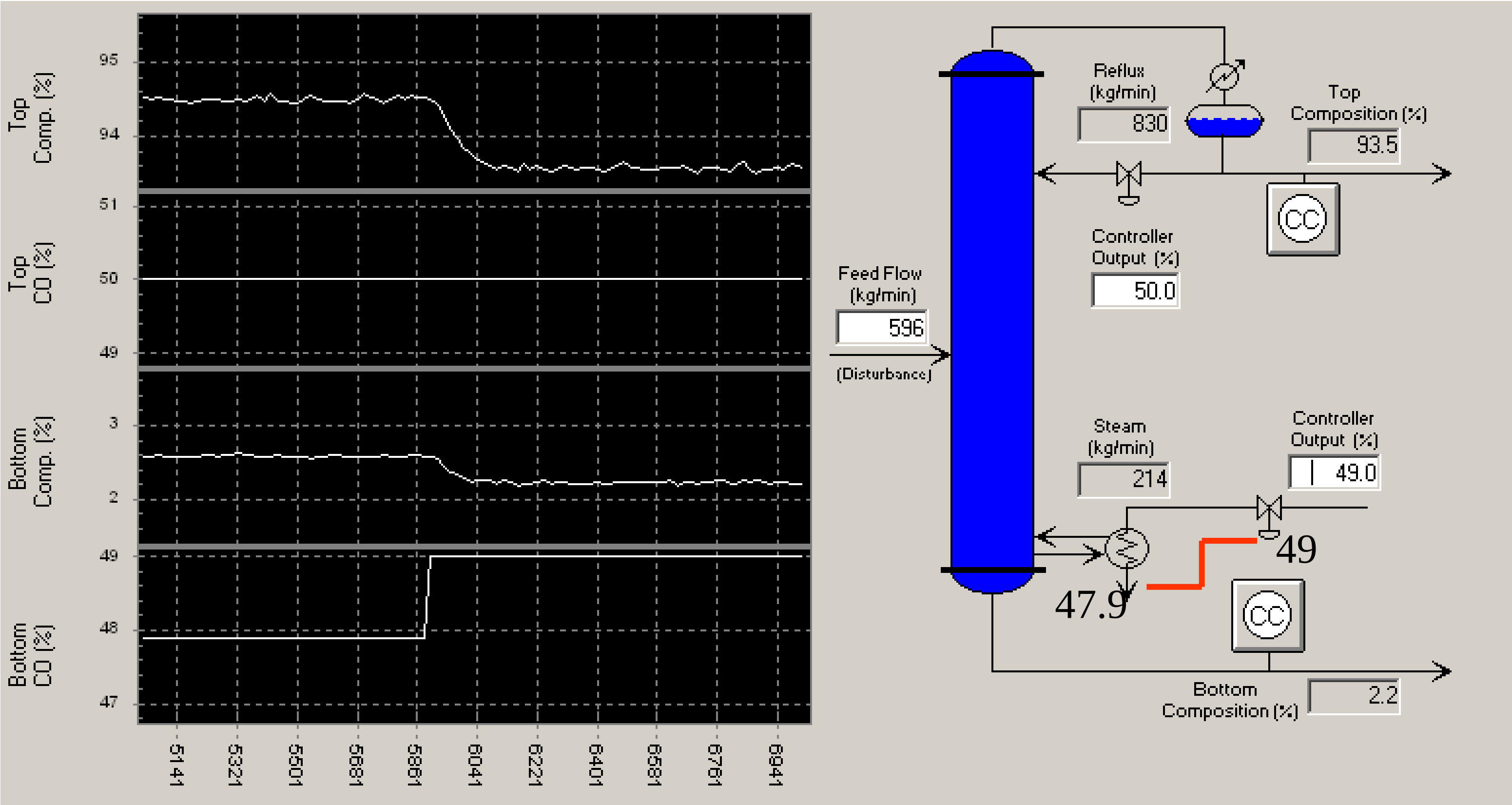


The models change because it is a non-linear system

$$G_{11} = \frac{K_{11}e^{-d_{11}s}}{\tau_{11}s+1} = \frac{0.936e^{-22.7s}}{72.8s+1}$$

$$G_{21} = \frac{K_{21}e^{-d_{21}s}}{\tau_{21}s+1} = \frac{0.470e^{-30.9s}}{66.65s+1}$$

Other MV

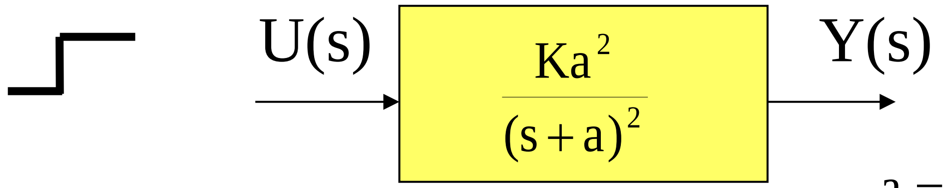


The models change because it is a non-linear system

$$G_{12} = \frac{K_{12}e^{-d_{12}s}}{\tau_{12}s+1} = \frac{-0.828e^{-22.36s}}{66.67s+1}$$

$$G_{22} = \frac{K_{22}e^{-d_{22}s}}{\tau_{22}s+1} = \frac{-0.345e^{-4.5s}}{57.02s+1}$$

Step response, $\delta = 1$



$$a = -\delta\omega_n$$

$$Y(s) = \frac{Ka^2}{(s+a)^2}\frac{u}{s} = \frac{\alpha}{s} + \frac{\beta}{s+a} + \frac{\gamma}{(s+a)^2} =$$

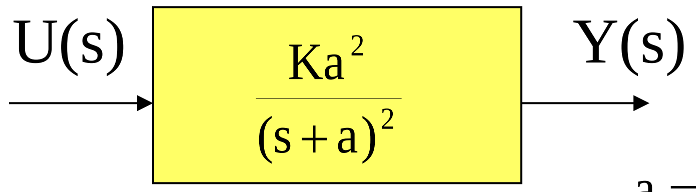
$$= \frac{\alpha(s+a)^2}{s(s+a)^2} + \frac{\beta s(s+a)}{s(s+a)^2} + \frac{\gamma s}{s(s+a)^2}$$

$$\text{for } s = 0 \quad \Rightarrow Ka^2u = \alpha a^2 \qquad \alpha = Ku$$

$$\text{for } s = -a \quad \Rightarrow Ka^2u = \gamma(-a) \qquad \gamma = -Kua = Ku\delta\omega_n$$

$$\text{for } s = a \quad \Rightarrow Ka^2u = Ku4a^2 + \beta 2a^2 - Kua^2 \qquad \beta = \text{-}Ku$$

Step response, $\delta = 1$



$a = -\delta\omega_n$

$$Y(s) = \left(\frac{\alpha}{s} + \frac{\beta}{s+a} + \frac{\gamma}{(s+a)^2}\right);$$

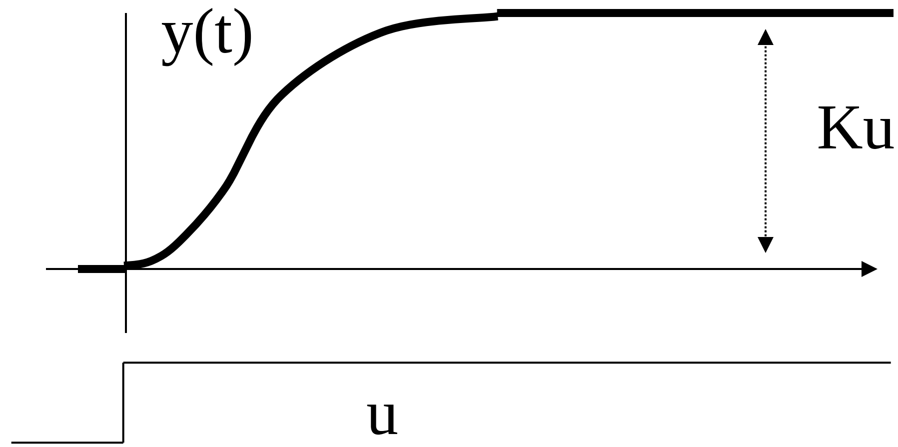
$$y(t) = L^{-1}\left[Y(s)\right] =$$

$$= L^{-1}\left[\frac{\alpha}{s}\right] + L^{-1}\left[\frac{\beta}{s+a}\right] + L^{-1}\left[\frac{\gamma}{(s+a)^2}\right]$$

$$y(t) = \alpha + \beta e^{-at} + \gamma t e^{-at} =$$

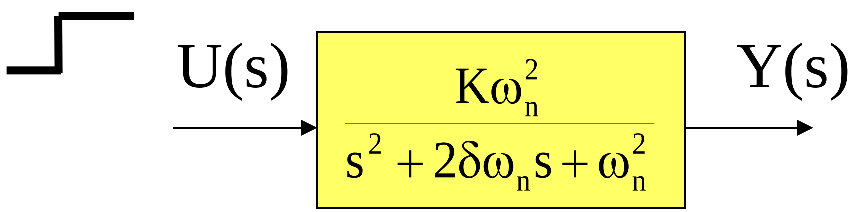
$$= Ku(1 - e^{-at} + \delta\omega_n t e^{-at})$$

$$y(0) = 0 \qquad y(\infty) = Ku$$



Monotonously increasing function

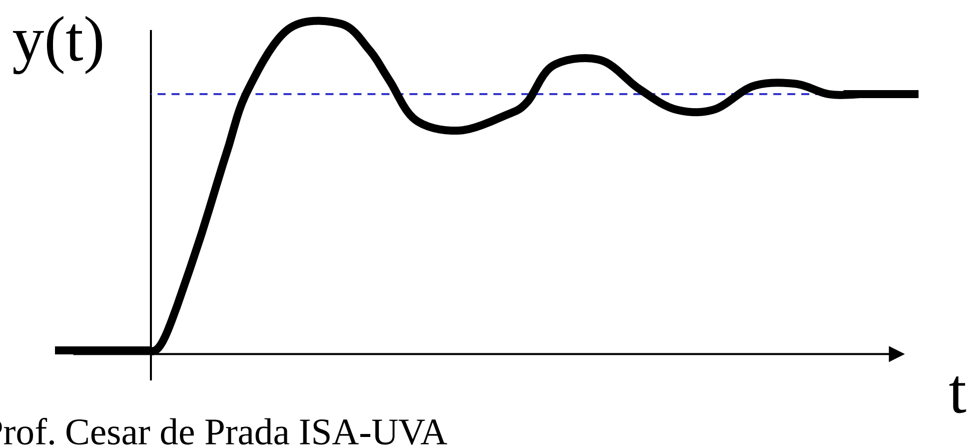
Step response, δ <1



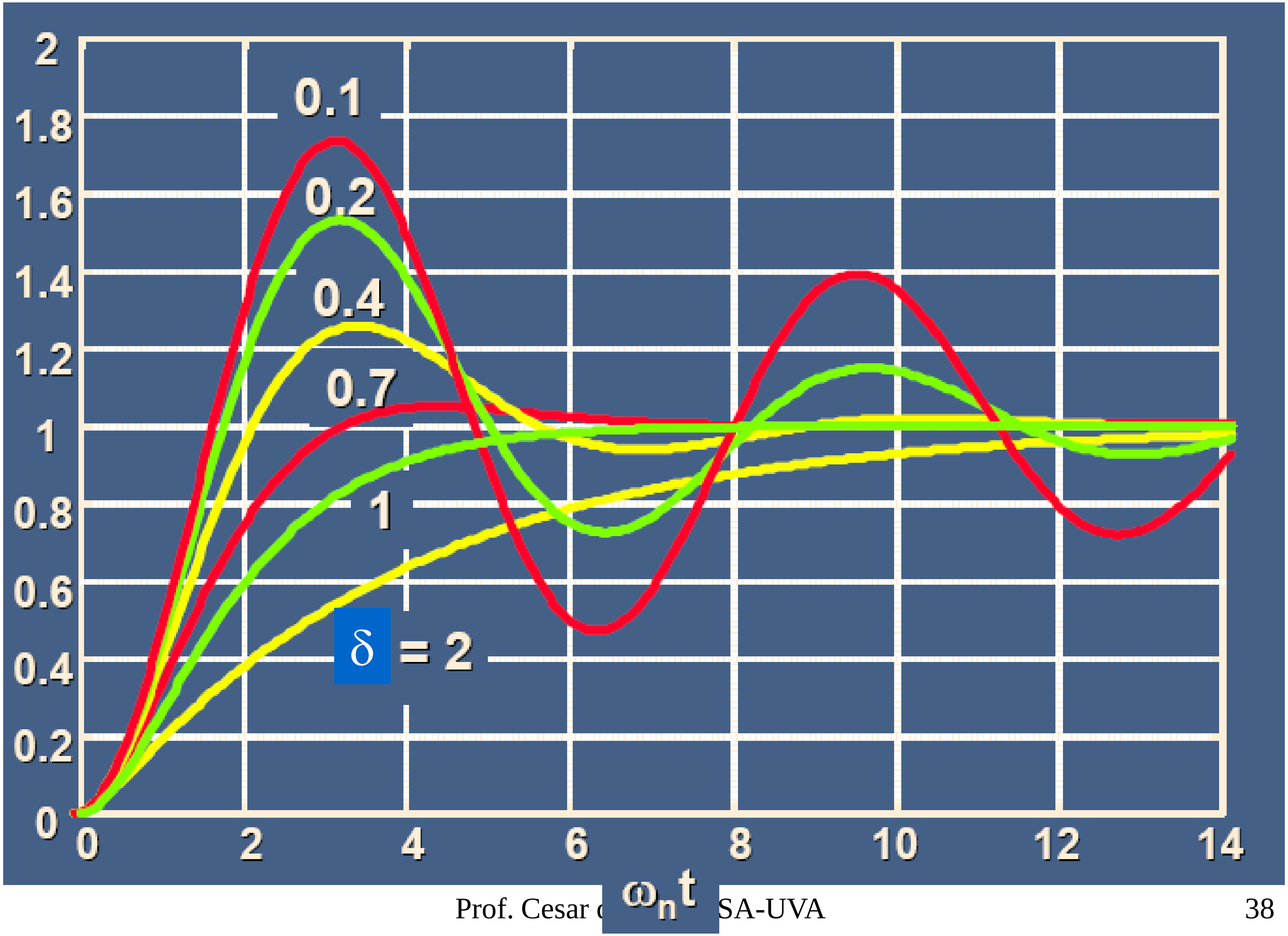
$$Y(s) = \frac{K\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}\frac{u}{s}$$

$$y(t) = \mathcal{L}^{-1}[Y(s)] = Ku\left[1 - \frac{1}{\sqrt{1-\delta^2}} e^{-\delta\omega_n t} \operatorname{sen}(\omega_n \sqrt{1-\delta^2}\, t + \phi)\right]$$

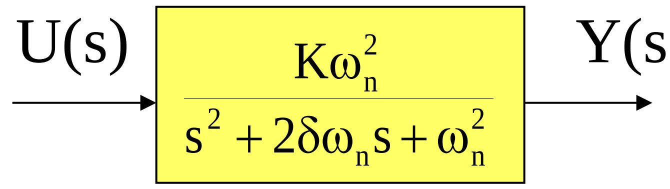
$$\phi = \operatorname{arctg}\frac{\sqrt{1-\delta^2}}{\delta}$$



If $\delta\omega_n > 0$: Time response stable, without delay and underdamped


2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0.1
0.2
0.4
0.7
1
δ = 2
0
2
4
6
8
10
12
14
ωnt

Step response, δ <1



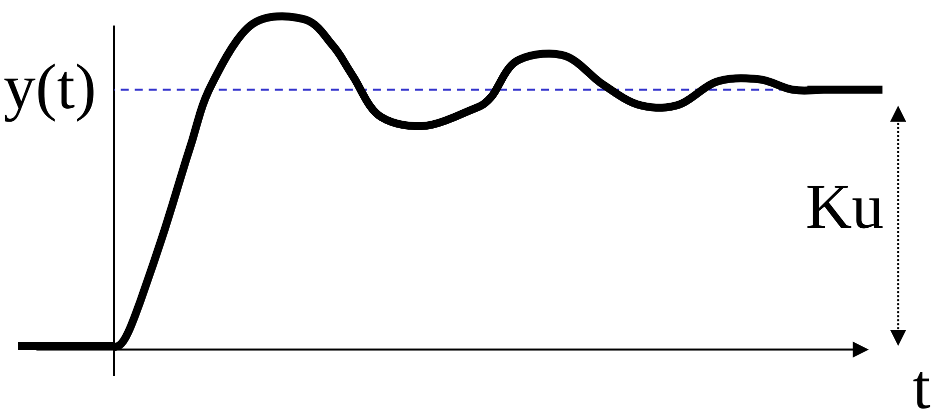
$$\frac{K\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}$$

$$y(t) = Ku\left[1 - \frac{1}{\sqrt{1-\delta^2}} e^{-\delta\omega_n t} \operatorname{sen}(\omega_n \sqrt{1-\delta^2}\, t + \phi)\right] \qquad \phi = \operatorname{arctg}\frac{\sqrt{1-\delta^2}}{\delta}$$

$y(0) = 0;$ $\quad y(\infty) = Ku;$ $\quad$ Gain : $Ku/u = K$

Oscillation frequency :

$$\omega_d = \omega_n \sqrt{1-\delta^2}$$



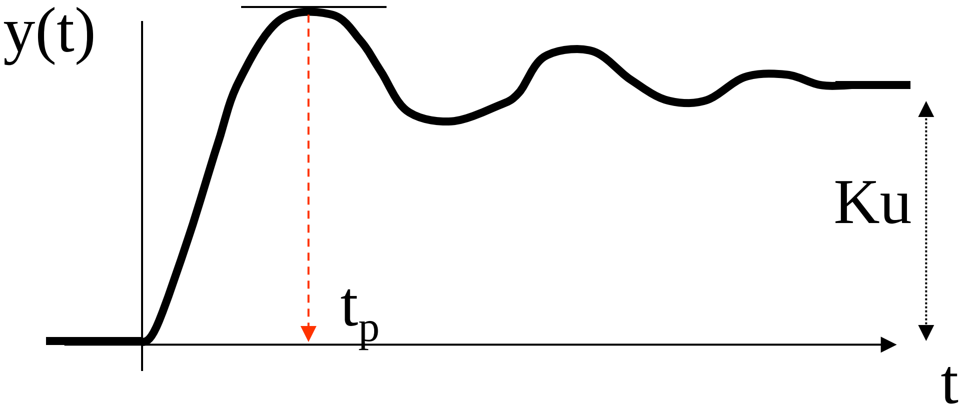
Peak Time

$$y(t) = Ku\left[1 - \frac{1}{\sqrt{1-\delta^2}} e^{-\delta\omega_n t} \mathrm{sen}(\omega_n \sqrt{1-\delta^2}\, t + \phi)\right]; \qquad \phi = \mathrm{arctg}\frac{\sqrt{1-\delta^2}}{\delta}$$

$$\left.\frac{d\,y(t)}{d\,t}\right|_{t=t_p} = 0$$

$$\frac{d\,y(t)}{d\,t} = \frac{-Ku}{\sqrt{1-\delta^2}}\left[-\delta\omega_n e^{-\delta\omega_n t} \mathrm{sen}(\omega_n \sqrt{1-\delta^2}\, t + \phi) + e^{-\delta\omega_n t} \cos(\omega_n \sqrt{1-\delta^2}\, t + \phi)\omega_n \sqrt{1-\delta^2}\right]$$

t_p = time to first peak

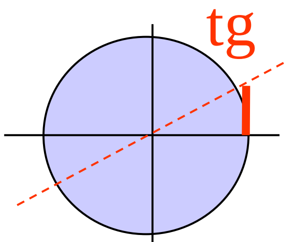


Peak Time

$$\frac{d\,y(t)}{d\,t} = \frac{-Ku}{\sqrt{1-\delta^2}}\left[-\delta\omega_n e^{-\delta\omega_n t}\,\text{sen}(\omega_n\sqrt{1-\delta^2}\,t+\phi) + e^{-\delta\omega_n t}\cos(\omega_n\sqrt{1-\delta^2}\,t+\phi)\,\omega_n\sqrt{1-\delta^2}\right]$$

$$\left.\frac{d\,y(t)}{d\,t}\right|_{t=t_p} = 0$$

$$\delta\omega_n e^{-\delta\omega_n t_p}\,\text{sen}(\omega_n\sqrt{1-\delta^2}\,t_p+\phi) = e^{-\delta\omega_n t_p}\cos(\omega_n\sqrt{1-\delta^2}\,t_p+\phi)\,\omega_n\sqrt{1-\delta^2}$$

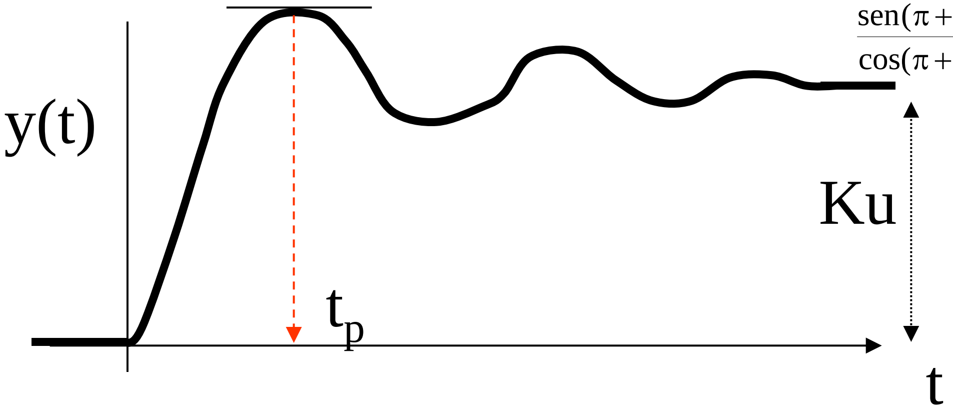


$$\frac{\text{sen}(\pi+\phi)}{\cos(\pi+\phi)} = \frac{-\text{sen}\phi}{-\cos\phi}$$

$$\text{tg}(\omega_n\sqrt{1-\delta^2}\,t_p+\phi) = \frac{\sqrt{1-\delta^2}}{\delta} = \text{tg}(\phi)$$

$$\omega_n\sqrt{1-\delta^2}\,t_p = \pm n\pi$$

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\delta^2}} = \frac{\pi}{\omega_d}$$



Percent overshoot

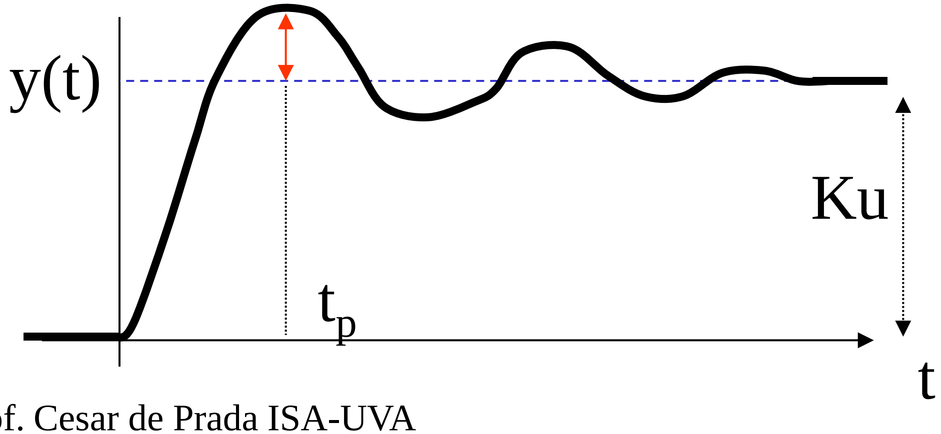
$$y(t) = Ku\left[1 - \frac{1}{\sqrt{1-\delta^2}} e^{-\delta\omega_n t} \operatorname{sen}(\omega_n \sqrt{1-\delta^2}\, t + \phi)\right] \qquad \phi = \operatorname{arctg}\frac{\sqrt{1-\delta^2}}{\delta}$$

$$M_p = \frac{y(t_p) - Ku}{Ku} 100 \text{ en } \% \qquad t_p = \frac{\pi}{\omega_n \sqrt{1-\delta^2}}$$

$$M_p = -\frac{100}{\sqrt{1-\delta^2}} e^{-\delta\omega_n \frac{\pi}{\omega_n\sqrt{1-\delta^2}}} \operatorname{sen}(\pi + \phi) = \frac{100}{\sqrt{1-\delta^2}} e^{-\frac{\pi\delta}{\sqrt{1-\delta^2}}} \operatorname{sen}(\phi) =$$

$$= \frac{100}{\sqrt{1-\delta^2}} e^{-\frac{\pi\delta}{\sqrt{1-\delta^2}}} \sqrt{1-\delta^2}$$

$$\boxed{M_p = 100 e^{-\frac{\pi\delta}{\sqrt{1-\delta^2}}} \text{ en } \%}$$



Settling time

$$y(t) = Ku\left[1 - \frac{1}{\sqrt{1-\delta^2}} e^{-\delta\omega_n t} \operatorname{sen}(\omega_n \sqrt{1-\delta^2}\, t + \phi)\right] \qquad \phi = \operatorname{arctg}\frac{\sqrt{1-\delta^2}}{\delta}$$

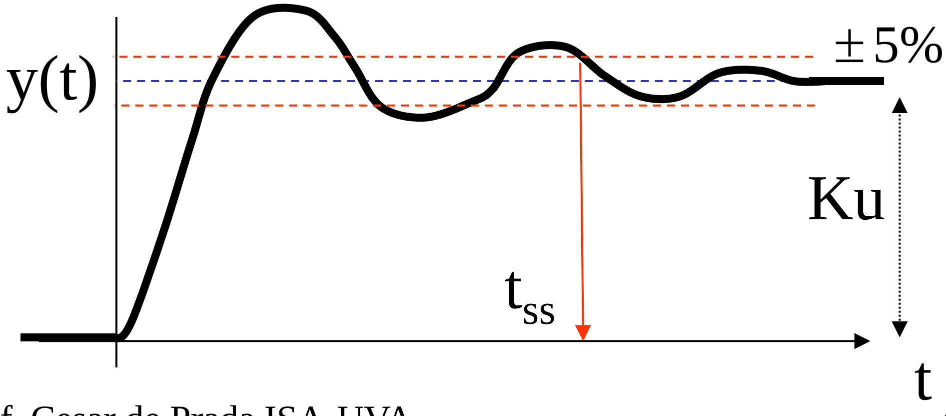
$$0.95Ku = Ku\left[1 - \frac{1}{\sqrt{1-\delta^2}} e^{-\delta\omega_n t_{ss}} \operatorname{sen}(\omega_n \sqrt{1-\delta^2}\, t_{ss} + \phi)\right]$$

$$\max\ t_{ss}\ \text{ such that } \left|\frac{1}{\sqrt{1-\delta^2}} e^{-\delta\omega_n t_{ss}} \operatorname{sen}(\omega_n \sqrt{1-\delta^2}\, t_{ss} + \phi)\right| = 0.05$$

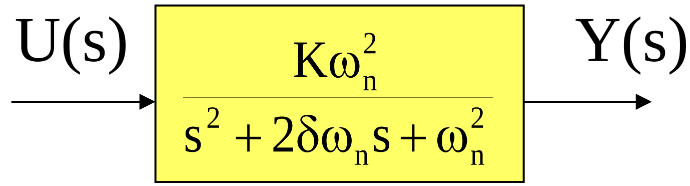
Implicit equation

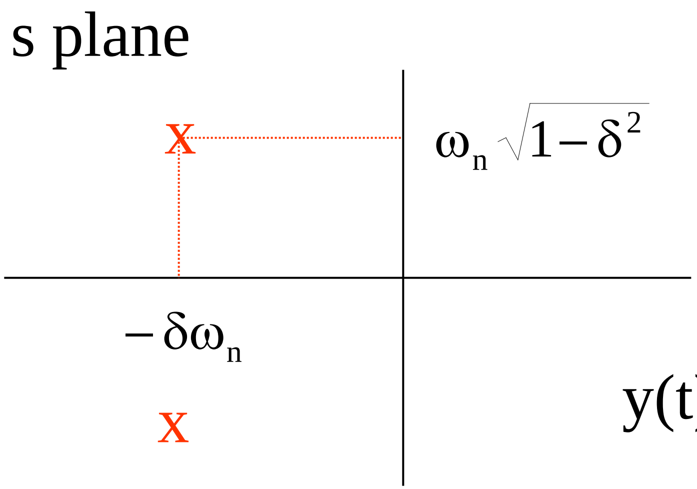
Approximately:

$$t_{ss} = \frac{3}{\delta\omega_n} \cdots \frac{5}{\delta\omega_n}$$

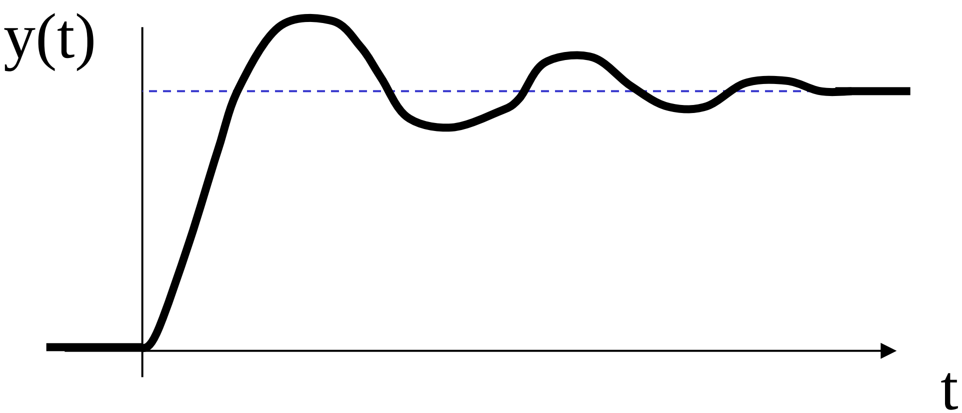


Interpretation in s





Polos: $-\delta\omega_n \pm j\omega_n \sqrt{1-\delta^2}$

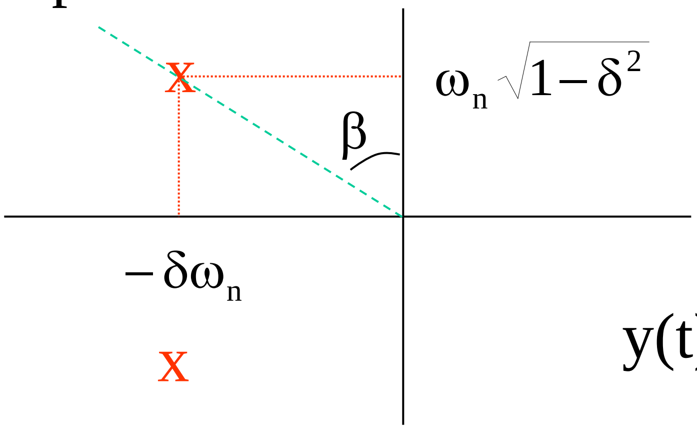


Complex conjugate poles located in the left hand side of the s plane

Interpretation in s

Poles: $$-\delta\omega_n \pm j\omega_n\sqrt{1-\delta^2}$$

s plane



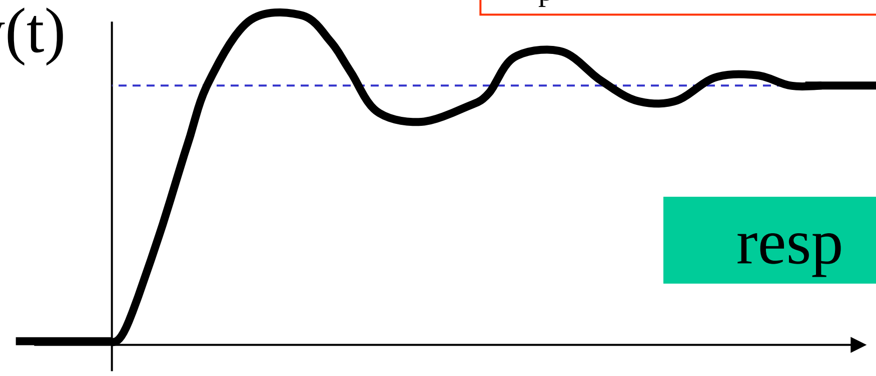
$$t_{ss} = \frac{3}{\delta\omega_n} \cdots \frac{5}{\delta\omega_n}$$

$$\omega_d = \omega_n\sqrt{1-\delta^2}$$

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\delta^2}} = \frac{\pi}{\omega_d}$$

$$tg(\beta) = \frac{\delta}{\sqrt{1-\delta^2}}$$

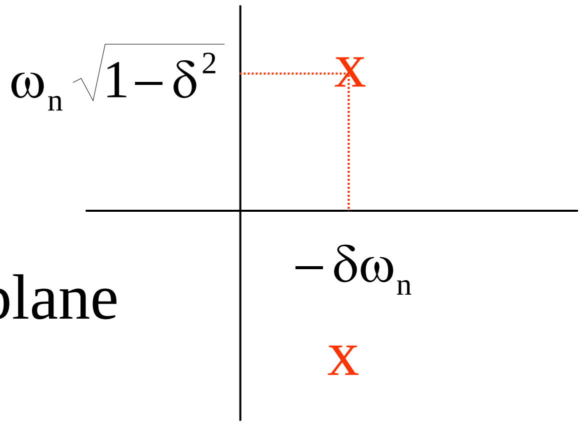
$$M_p = 100e^{-\frac{\pi\delta}{\sqrt{1-\delta^2}}} \text{ en } \%$$



resp

Complex conjugate poles located in the left hand side of the s plane

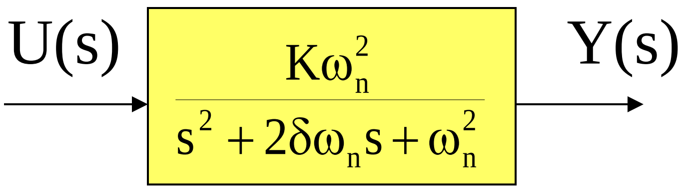
Interpretation in s

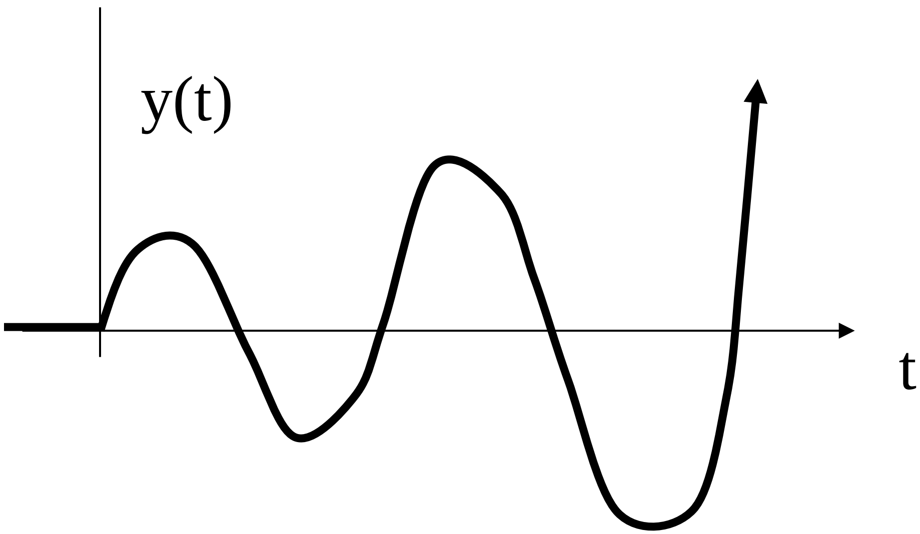


s plane

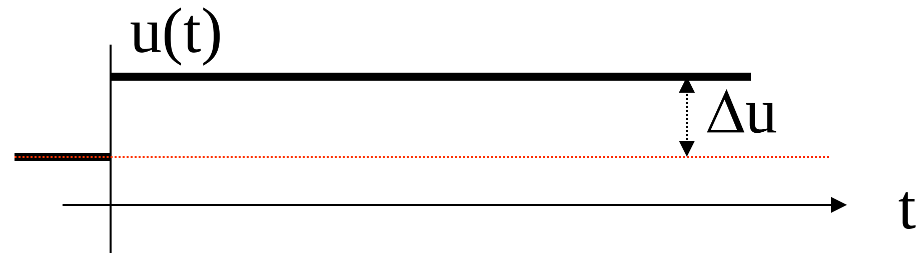
$$y(t) = Ku\left[1 - \frac{1}{\sqrt{1-\delta^2}} e^{-\delta\omega_n t} \operatorname{sen}(\omega_n\sqrt{1-\delta^2}\, t + \phi)\right]$$

si $\quad \delta\omega_n < 0 \quad$ Unstable system

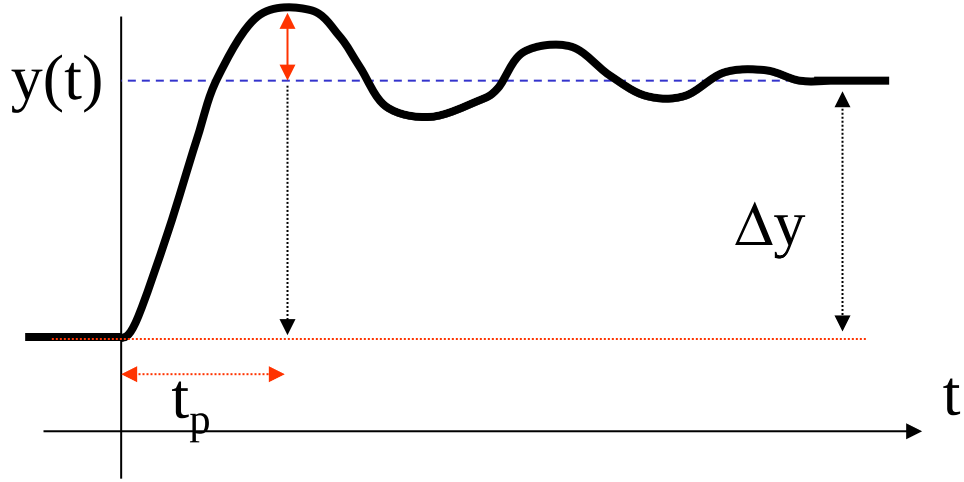




Identification



If the time response to a input step Δu starting from an equilibrium point is like the one in the figure ⇒ second order system with complex conjugate poles



$$\frac{K\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}$$

Parameter estimation:

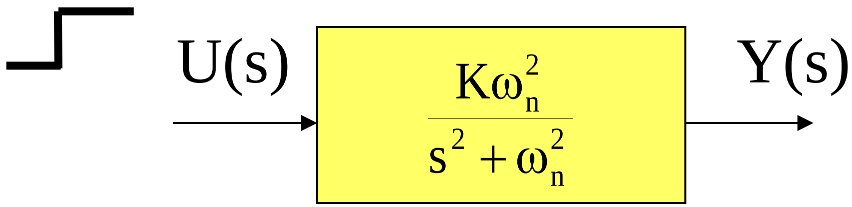
$K = \Delta y / \Delta u$

$$M_p = 100 e^{-\frac{\pi\delta}{\sqrt{1-\delta^2}}} \text{ en } \%$$

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\delta^2}} = \frac{\pi}{\omega_d}$$

Problema56

Step response, δ =0

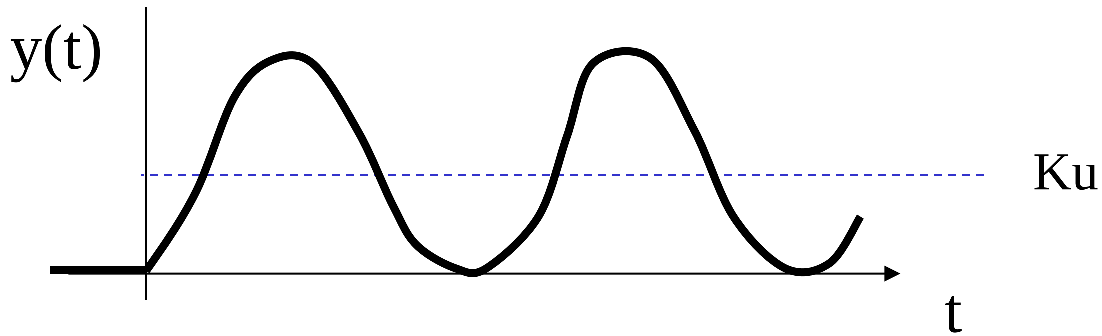


$$Y(s) = \frac{K\omega_n^2}{s^2+\omega_n^2}\frac{u}{s}$$

$$y(t) = L^{-1}[Y(s)] = Ku\left[1 - \text{sen}(\omega_n t + \frac{\pi}{2})\right]$$

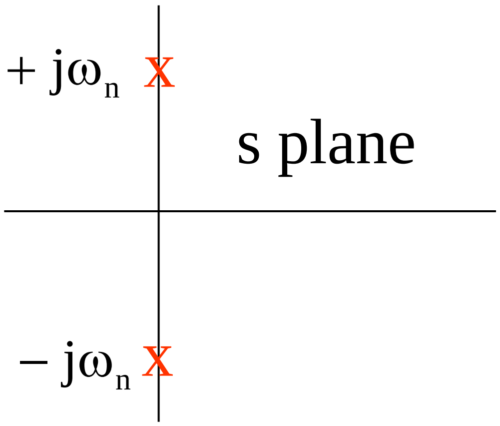
Undamped system

As δ = 0, the time response never damps. Time response in the stability border

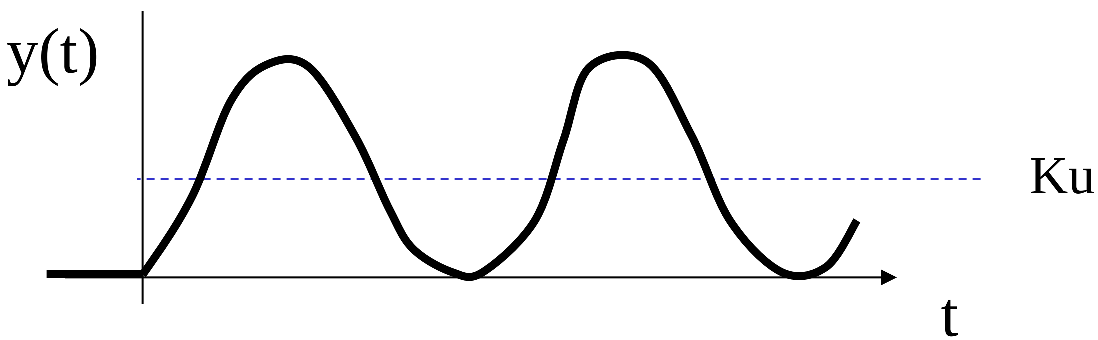


Interpretation in s

$$\frac{K\omega_n^2}{s^2+\omega_n^2} \qquad \text{poles:} \quad s^2+\omega_n^2=0 \quad \Rightarrow \quad s=\pm j\omega_n$$



$$y(t)=L^{-1}\left[Y(s)\right]=Ku\left[1-\text{sen}(\omega_n t+\frac{\pi}{2})\right]$$

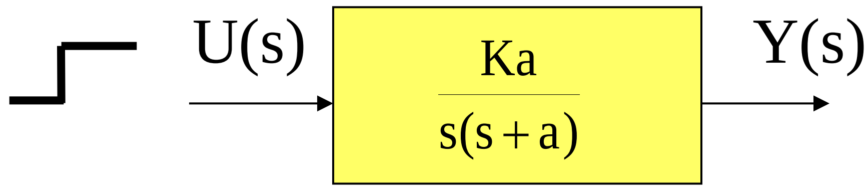


Poles located on the imaginary axis: stability border

resp

SysQuake

Poles at the origin: Integrators



$$Y(s) = \frac{Ka}{(s+a)s}\frac{u}{s} = \frac{\alpha}{s} + \frac{\beta}{s^2} + \frac{\gamma}{s+a} =$$

$$= \frac{\alpha s(s+a)}{s^2(s+a)} + \frac{\beta(s+a)}{s^2(s+a)} + \frac{\gamma s^2}{s^2(s+a)}$$

Step response

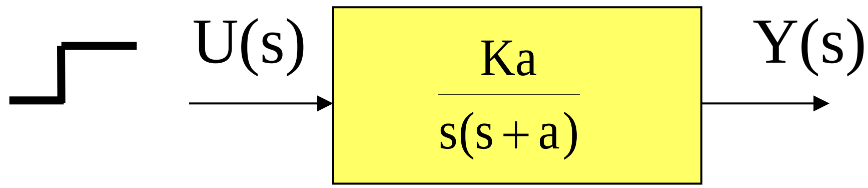
$$\text{for } s = 0 \quad \Rightarrow Kau = \beta a \qquad \beta = Ku$$

$$\text{for } s = -a \quad \Rightarrow Kau = \gamma a^2 \qquad \gamma = Ku/a$$

$$\text{for } s = a \quad \Rightarrow Kau = \alpha 2a^2 + \beta 2a + \gamma a^2$$

$$Ku = \alpha 2a + 2Ku + Ku \quad \Rightarrow \quad \alpha = \text{-}Ku/a$$

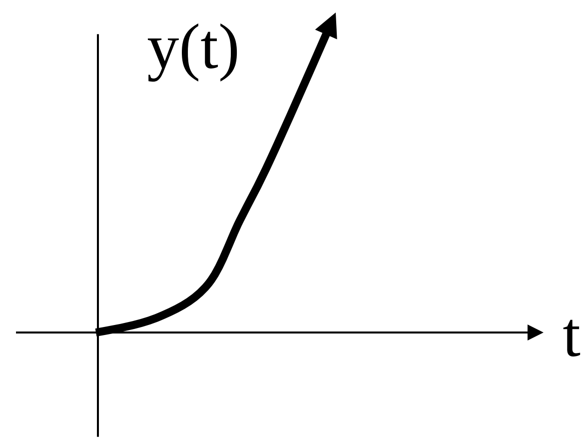
Poles at the origin: Integrators

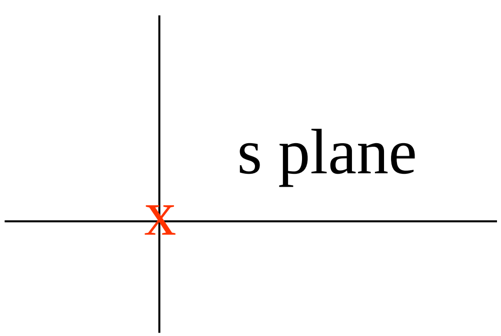


$$Y(s) = \frac{Ka}{(s+a)s}\frac{u}{s} = \frac{\alpha}{s} + \frac{\beta}{s^2} + \frac{\gamma}{s+a}$$

$$y(t) = L^{-1}[Y(s)] = L^{-1}\left[\frac{\alpha}{s}\right] + L^{-1}\left[\frac{\beta}{s^2}\right] + L^{-1}\left[\frac{\gamma}{s+a}\right]$$

$$y(t) = \alpha + \beta t + \gamma e^{-at} = Ku\left[\frac{1}{a} + t - \frac{1}{a}e^{-at}\right]$$



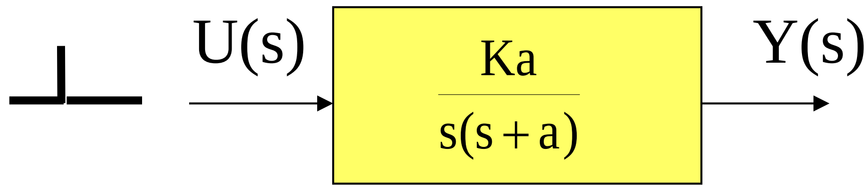


respx

SysQuake

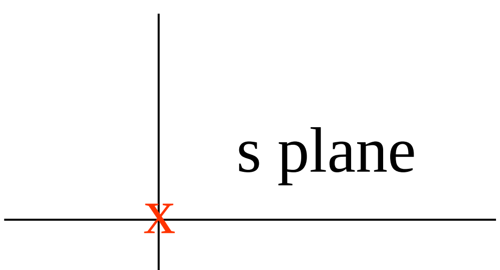
Poles at the origin: Integrators

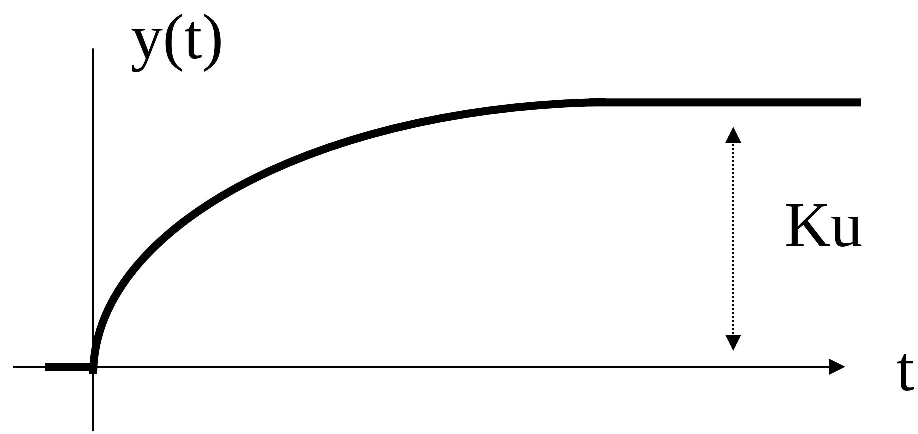
Another input:
impulse u



$$Y(s) = \frac{Ka}{(s+a)s}u$$

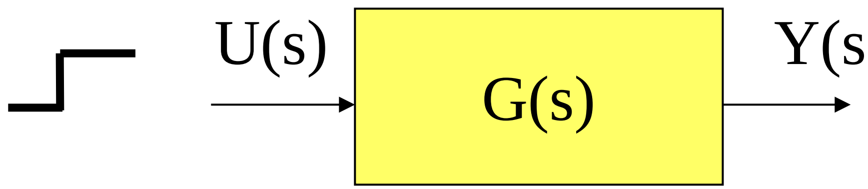
$$y(t) = L^{-1}[Y(s)] = Ku\left[1 - e^{-at}\right]$$





The steady state is related to the integral of the input, so BIBO stability depends on the type of input

Time response of higher order systems



respx

$$Y(s) = \left(\frac{\alpha}{s} + \frac{\beta}{s+a} + \frac{\gamma}{s+b} + \frac{\upsilon}{(s+b)^2} + \ldots + \frac{\sigma}{s^2 + 2\delta\omega_n s + \omega_n^2} + \ldots\right);$$

$$y(t) = L^{-1}\left[Y(s)\right] =$$

$$= L^{-1}\left[\frac{\alpha}{s}\right] + L^{-1}\left[\frac{\beta}{s+a}\right] + L^{-1}\left[\frac{\gamma}{s+b}\right] + L^{-1}\left[\frac{\upsilon}{(s+b)^2}\right] + \ldots + L^{-1}\left[\frac{\sigma}{s^2 + 2\delta\omega_n s + \omega_n^2}\right] + \ldots$$

$$y(t) = \alpha + \beta e^{-at} + \gamma e^{-bt} + \upsilon t e^{-bt} + \ldots + e^{-\delta\omega_n t}\,\mathrm{sen}(\omega_n\sqrt{1-\delta^2} + \phi) + \ldots$$

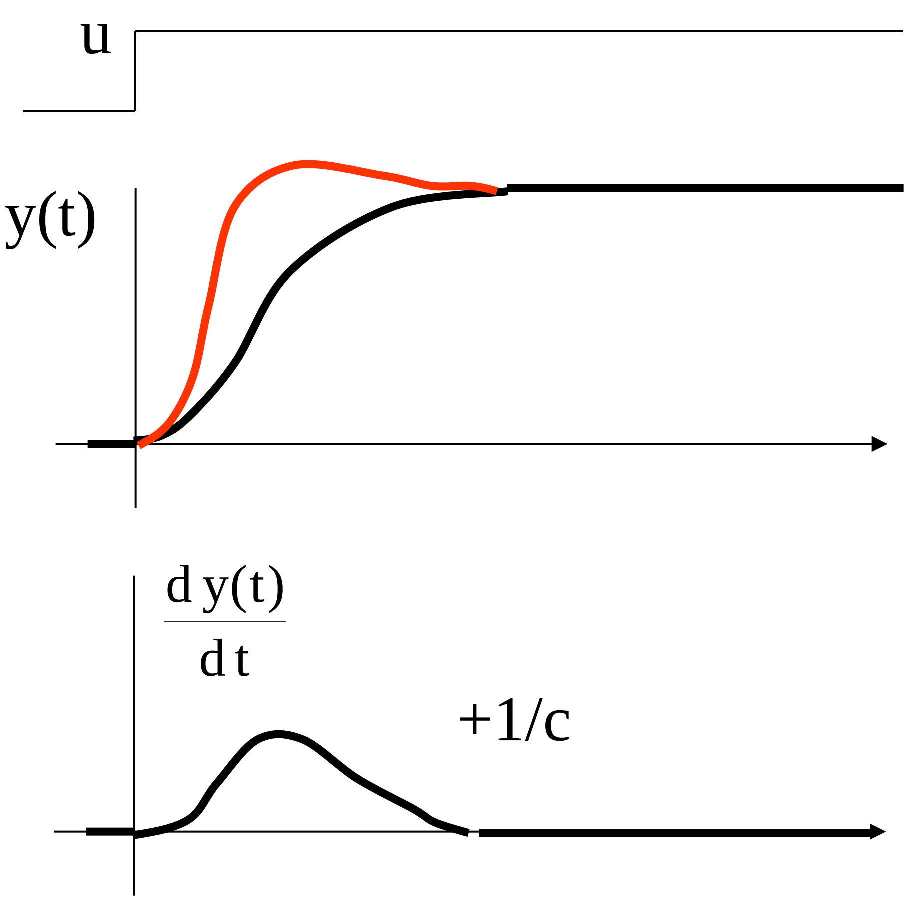
Poles of G(s) determine the stability and the type of time response. Zeros of G(s) may modify the shape of the response but not the stability

How a zero modify the time response

$$G(s)\left(\frac{1}{c}s+1\right) = G(s) + \frac{1}{c}sG(s)$$

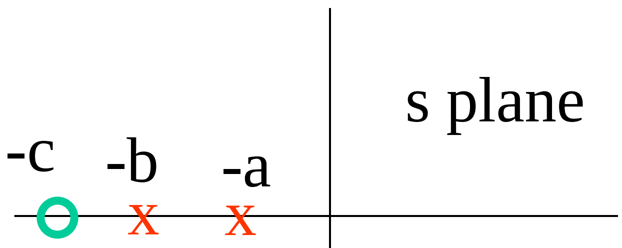
The time response against the same input of a system with an additional zero at s = - c, can be obtained adding to the original (non zero) response its derivative times a factor 1/c

Zeros in the left half plane



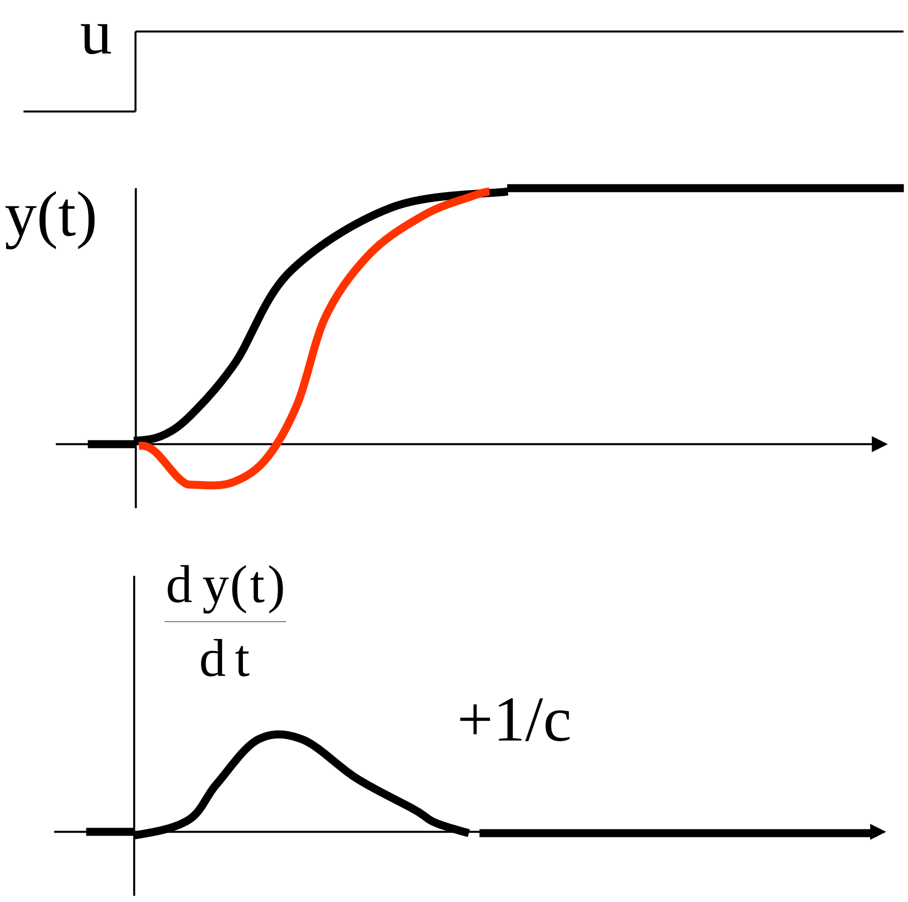
With c > 0, the time response is moved forward.

The zero does not create oscillations but can produce an overshoot

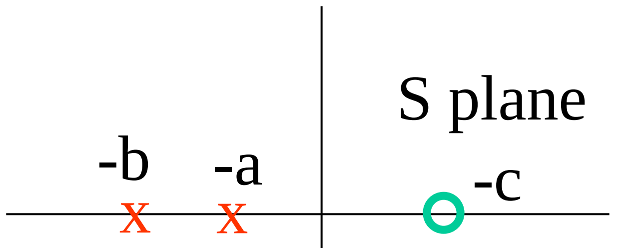


Zero located on the real axis in the left half plane

Zeros on the right half plane

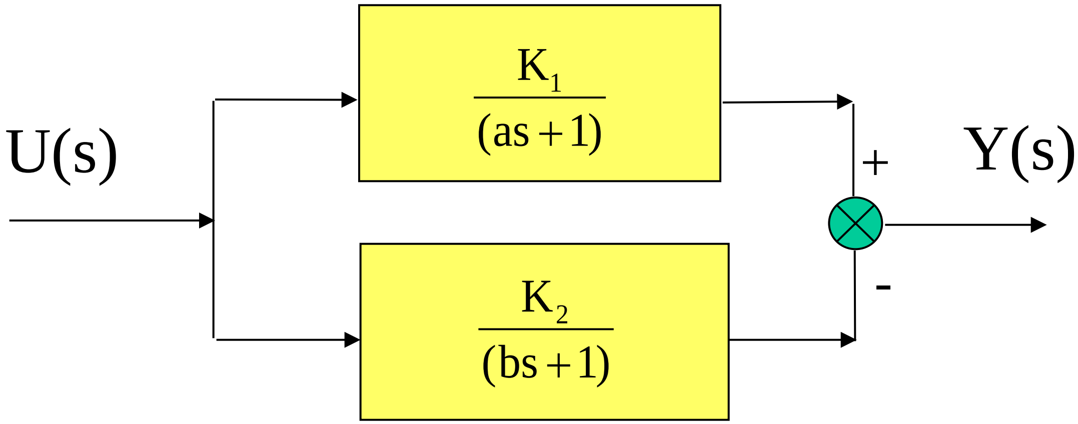


with c < 0, the time response create an inverse response (non-minimum phase)



Zero located on the real axis, in the right half plane

Interpretation of a zero



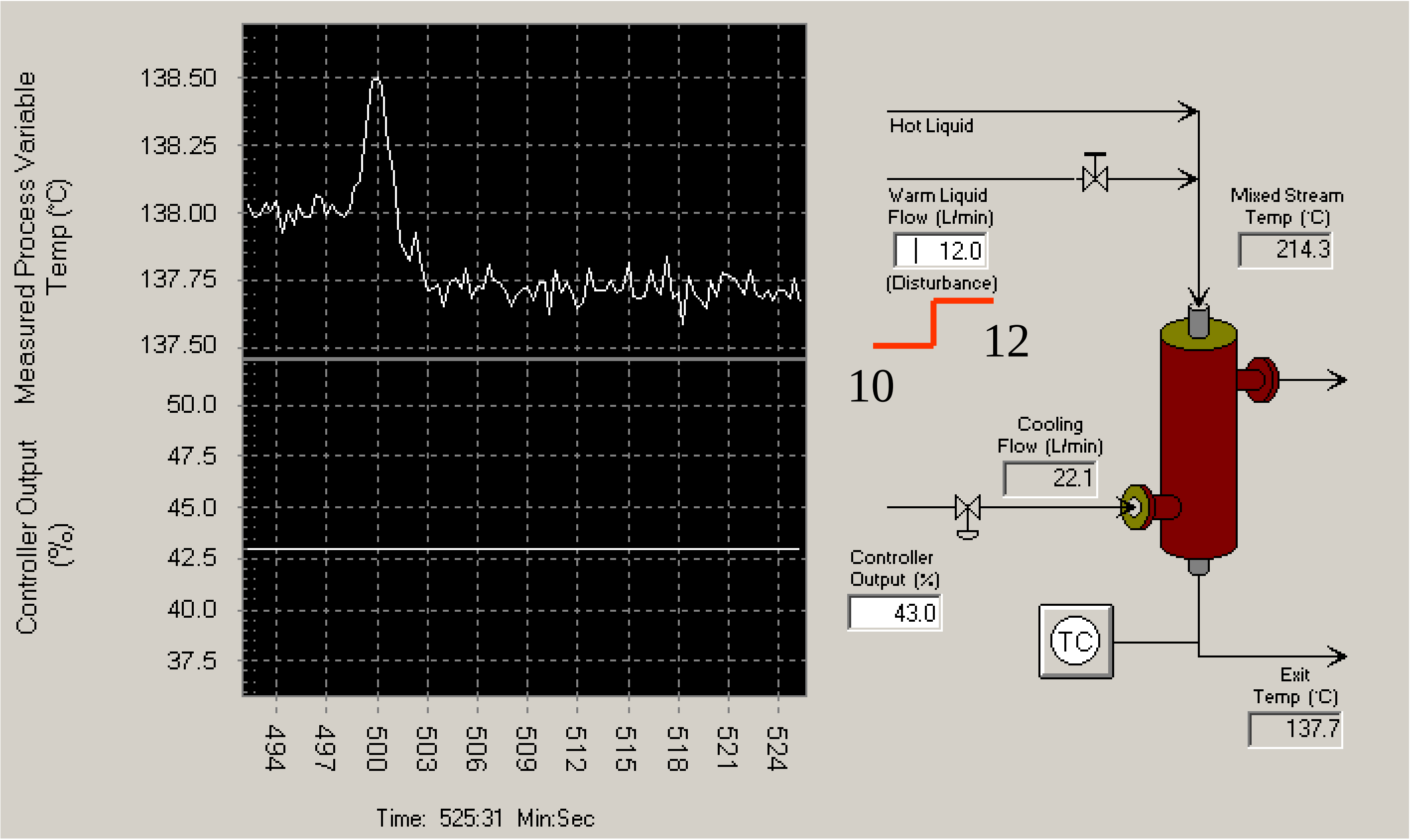
respcero

$$Y(s) = \left[\frac{K_1}{(as+1)} - \frac{K_2}{(bs+1)}\right]U(s) = \frac{K_1(bs+1) - K_2(as+1)}{(as+1)(bs+1)}U(s) = \frac{(K_1 b - K_2 a)s + (K_1 - K_2)}{(as+1)(bs+1)}U(s)$$

A zero appears when the same cause creates two different additive effects on the output variable. If these effects have opposite signs, then the zero is located on the right half plane

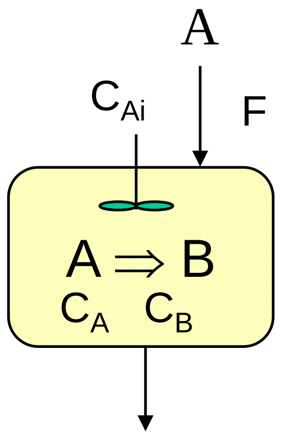
Heat exchanger

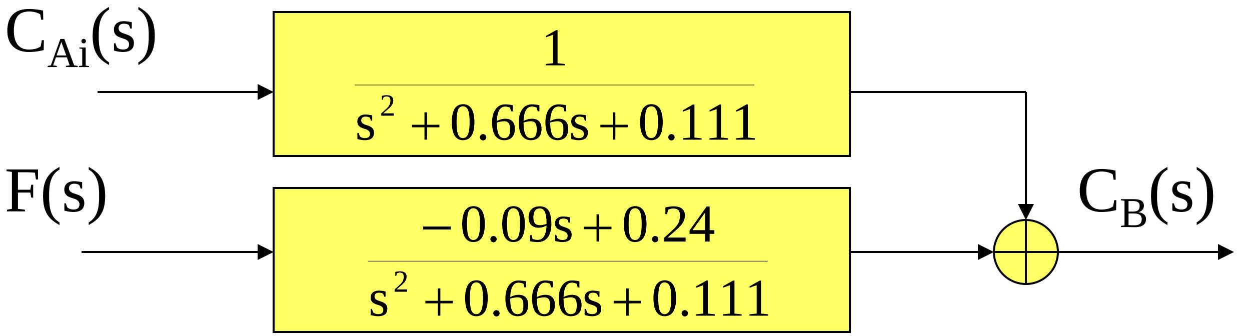
Open loop test



Isothermal Reactor

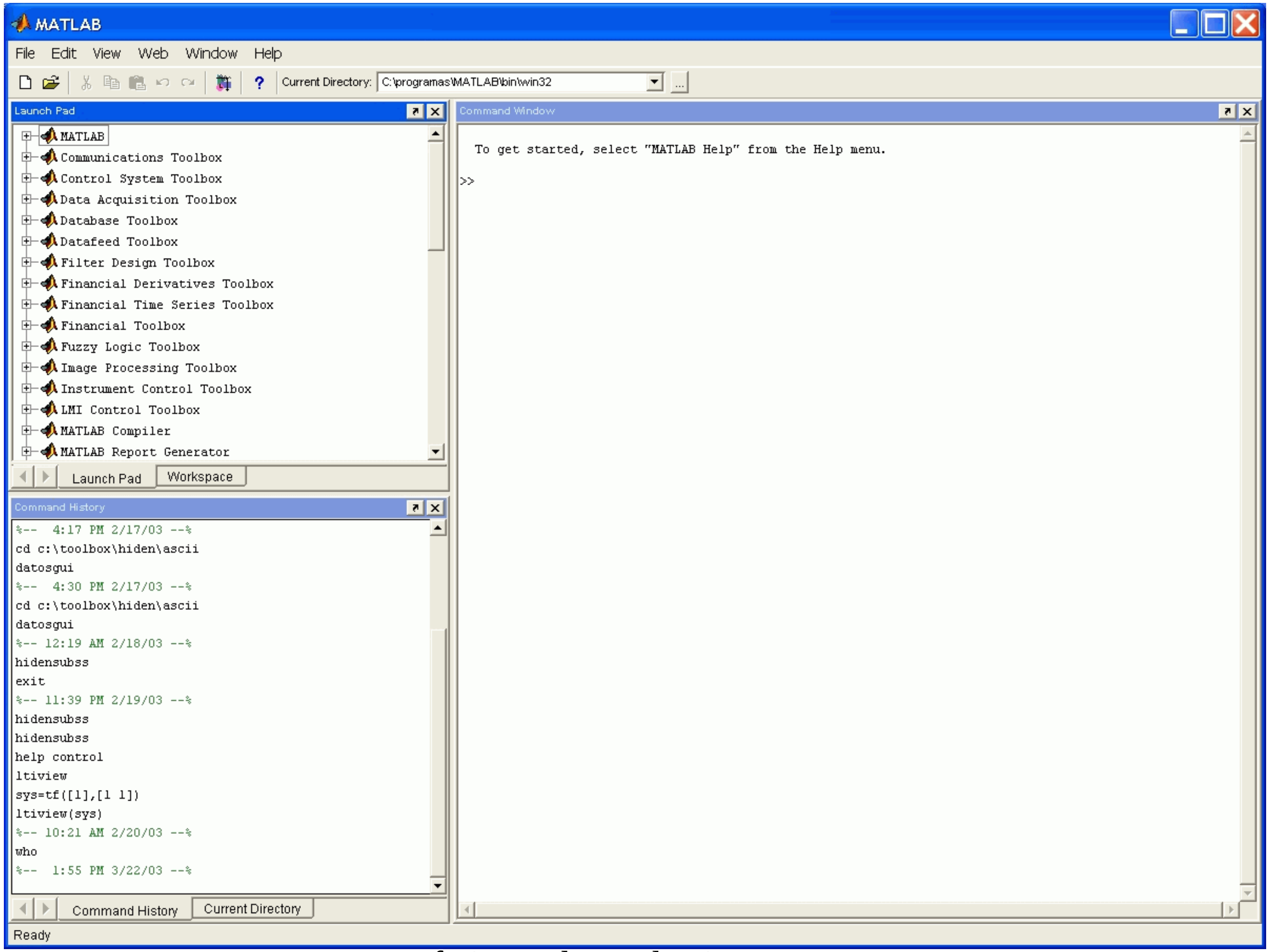
$$\begin{bmatrix} \frac{d\Delta c_A}{dt} \\ \frac{d\Delta c_B}{dt} \end{bmatrix} = \begin{pmatrix} -0.33 & 0 \\ 3 & -0.33 \end{pmatrix} \begin{bmatrix} \Delta c_A \\ \Delta c_B \end{bmatrix} + \begin{pmatrix} 0.09 & 0.333 \\ -0.09 & 0 \end{pmatrix} \begin{bmatrix} \Delta F \\ \Delta c_{Ai} \end{bmatrix}$$



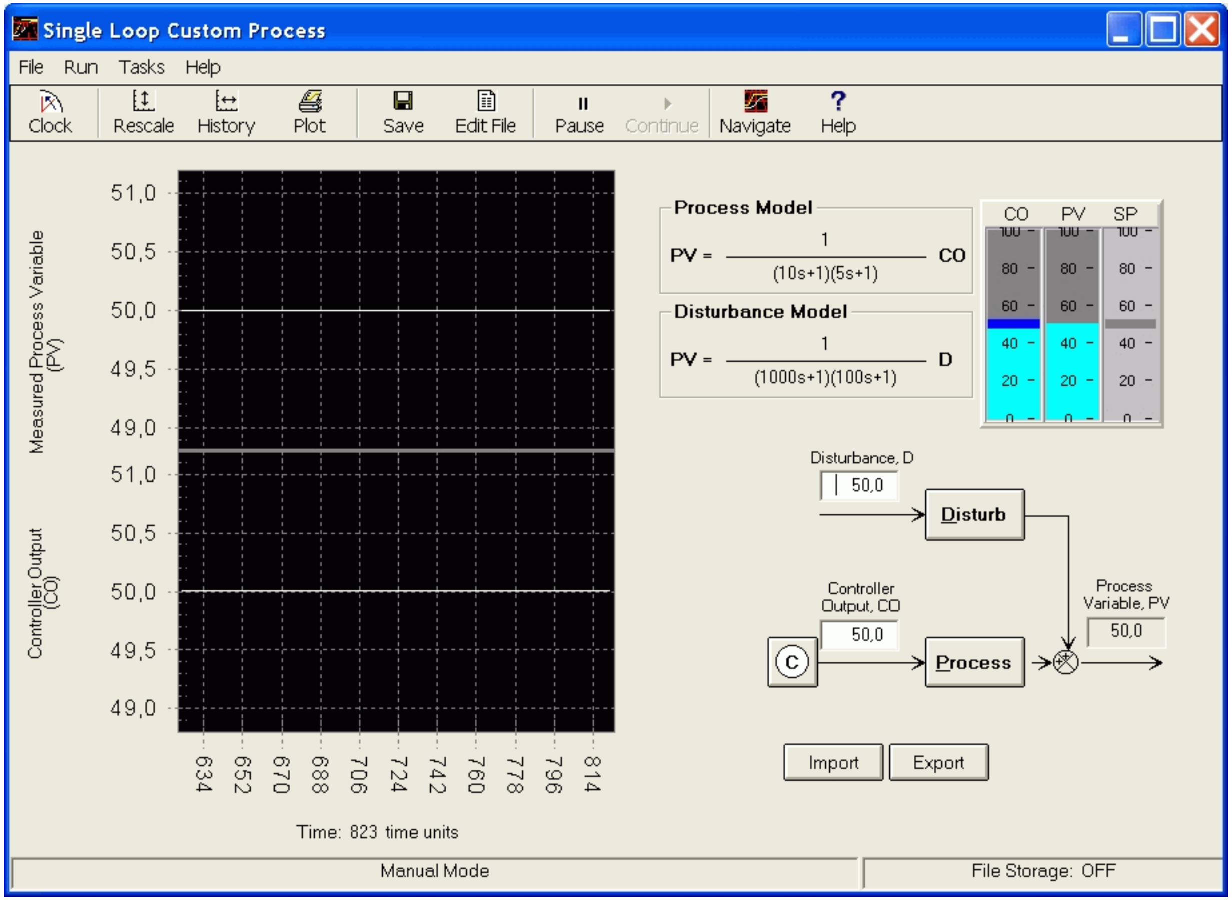


-0.3330 + 0.0105i
-0.3330 - 0.0105i

Matlab

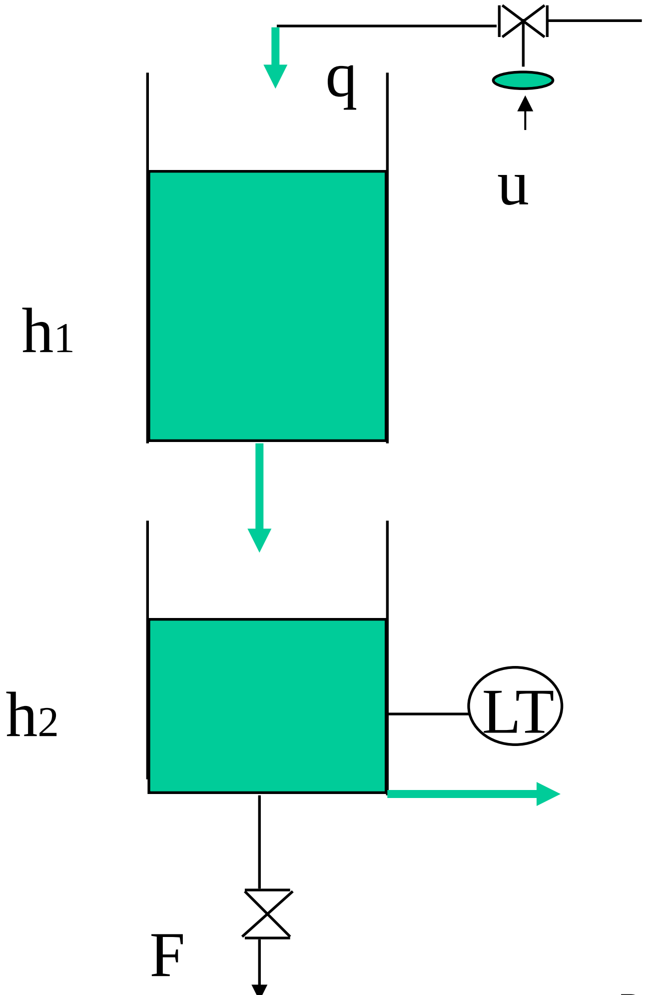

MATLAB
File Edit View Web Window Help
Current Directory: C:\programas\MATLAB\bin\win32
Launch Pad
MATLAB
Communications Toolbox
Control System Toolbox
Data Acquisition Toolbox
Database Toolbox
Datafeed Toolbox
Filter Design Toolbox
Financial Derivatives Toolbox
Financial Time Series Toolbox
Financial Toolbox
Fuzzy Logic Toolbox
Image Processing Toolbox
Instrument Control Toolbox
LMI Control Toolbox
MATLAB Compiler
MATLAB Report Generator
Launch Pad
Workspace
Command History
%-- 4:17 PM 2/17/03 --%
cd c:\toolbox\hiden\ascii
datosgui
%-- 4:30 PM 2/17/03 --%
cd c:\toolbox\hiden\ascii
datosgui
%-- 12:19 AM 2/18/03 --%
hidensubss
exit
%-- 11:39 PM 2/19/03 --%
hidensubss
hidensubss
help control
ltiview
sys=tf([1],[1 1])
ltiview(sys)
%-- 10:21 AM 2/20/03 --%
who
%-- 1:55 PM 3/22/03 --%
Command History
Current Directory
Command Window
To get started, select "MATLAB Help" from the Help menu.
>>
Ready

Cstation



Single Loop Custom Process
File Run Tasks Help
Clock Rescale History Plot Save Edit File Pause Continue Navigate Help
Measured Process Variable (PV)
Controller Output (CO)
51,0 50,5 50,0 49,5 49,0
634 652 670 688 706 724 742 760 778 796 814
Time: 823 time units
Process Model
PV = 1 / (10s+1)(5s+1) CO
Disturbance Model
PV = 1 / (1000s+1)(100s+1) D
CO PV SP
Disturbance, D
50,0
Disturb
Controller Output, CO
50,0
Process
Process Variable, PV
50,0
Import Export
Manual Mode
File Storage: OFF

Two tanks

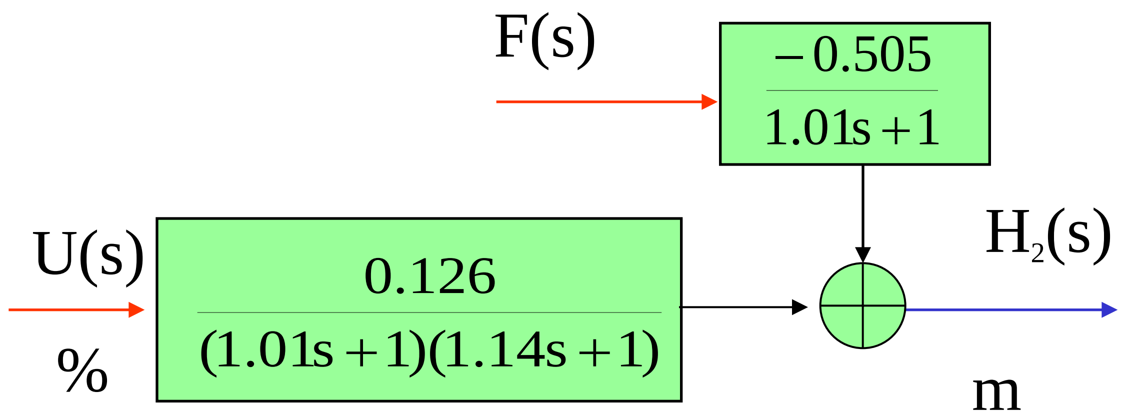


Operating point:

q=17.8 l/m u= 70 %

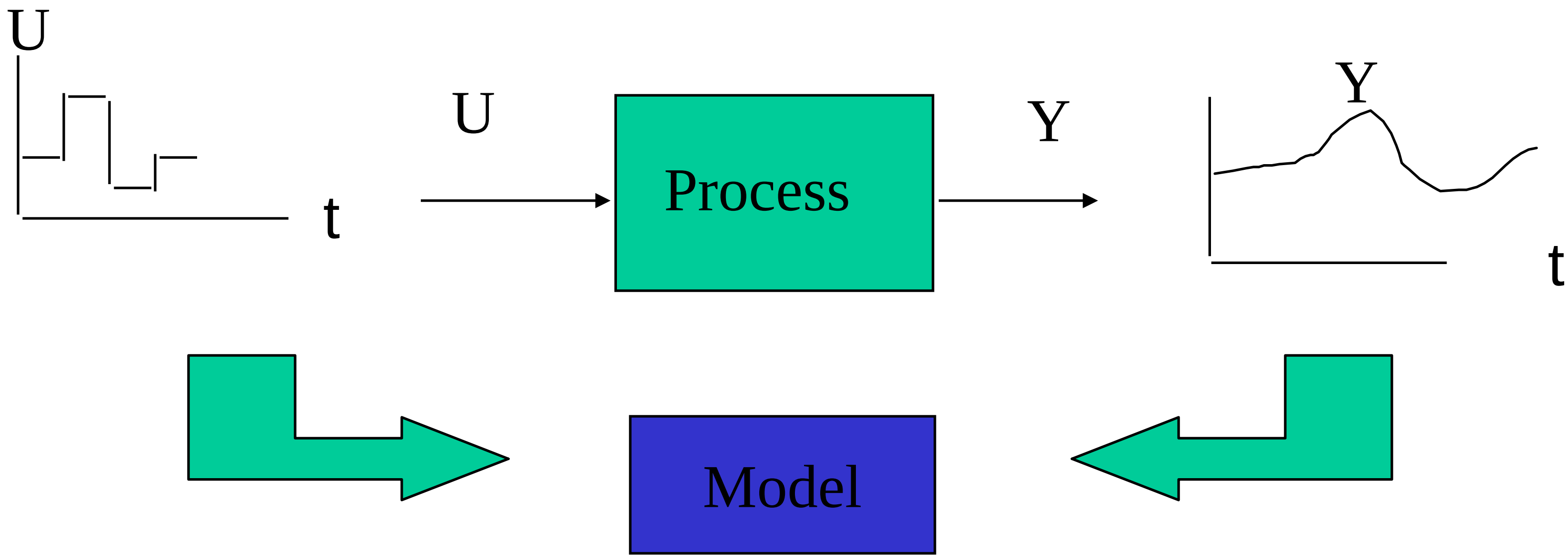
F=2 l/m h_{20}= 4 m

A_1=0.2 dm^2 A_1=0.2 dm^2



Identification

The model is obtained from input-output experimental data



Identification methodology

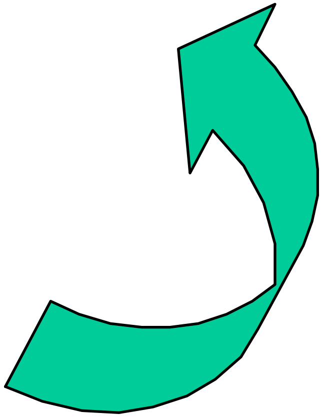
Process knowledge and experiment design

Experiments and data collection

Analysis and data processing

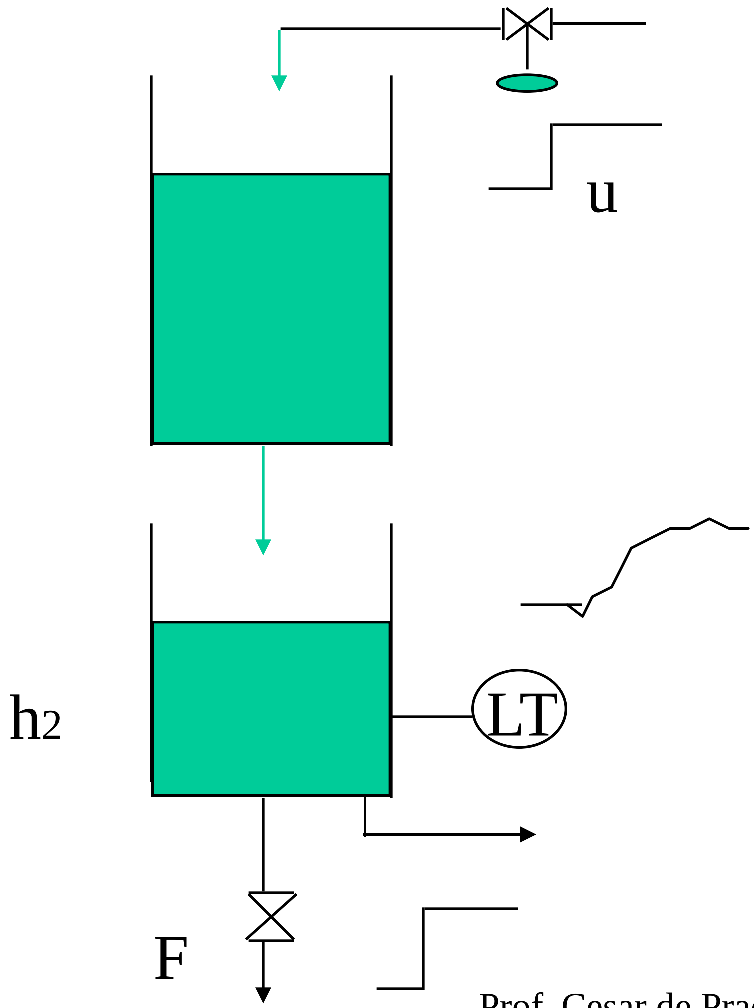
Selection of model class

Parameter estimation



Model validation

Step response identification



Two experiments:

- Step change in u, F = cte.
- Step change in F, u = cte.

Model class chosen:

First order and first order plus delay functions

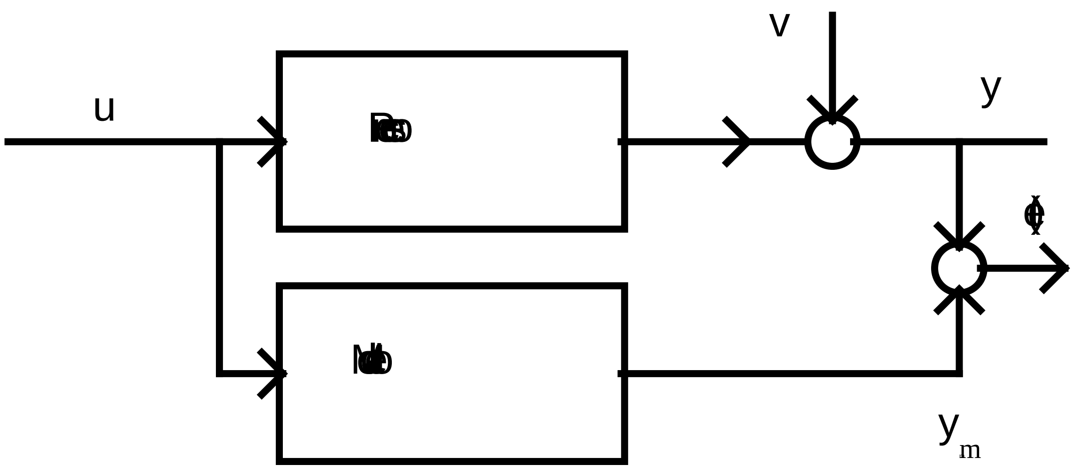
$$H_2(s) = \frac{K_q e^{-ds}}{\tau_q s + 1} U(s) = \frac{0.127 e^{-0.71s}}{1.64s + 1} U(s)$$

$$H_2(s) = \frac{K_f}{\tau_f s + 1} F(s) = \frac{-0.5}{0.99s + 1} F(s)$$

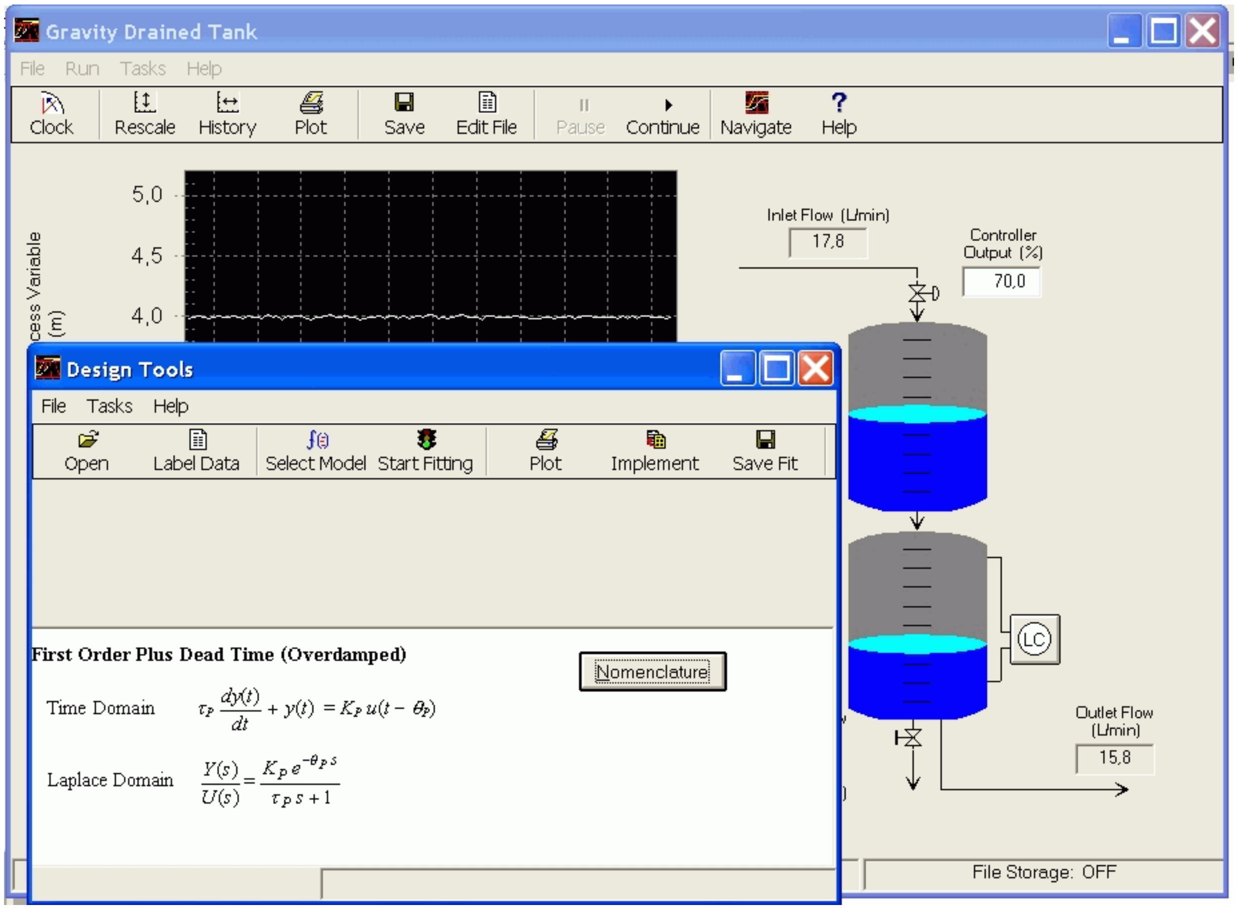
Least Squares (LS)

Identification criterium: Given a set of experimental data u(t), y(t), t = 1,2,3,…,N, find the model parameters, θ, that minimize the cost function V :

$$V = \frac{1}{N}\sum_{t=1}^{N} e(t)^2 = \frac{1}{N}\sum_{t=1}^{N}\left[(y(t) - y_m(t,\theta)\right]^2$$



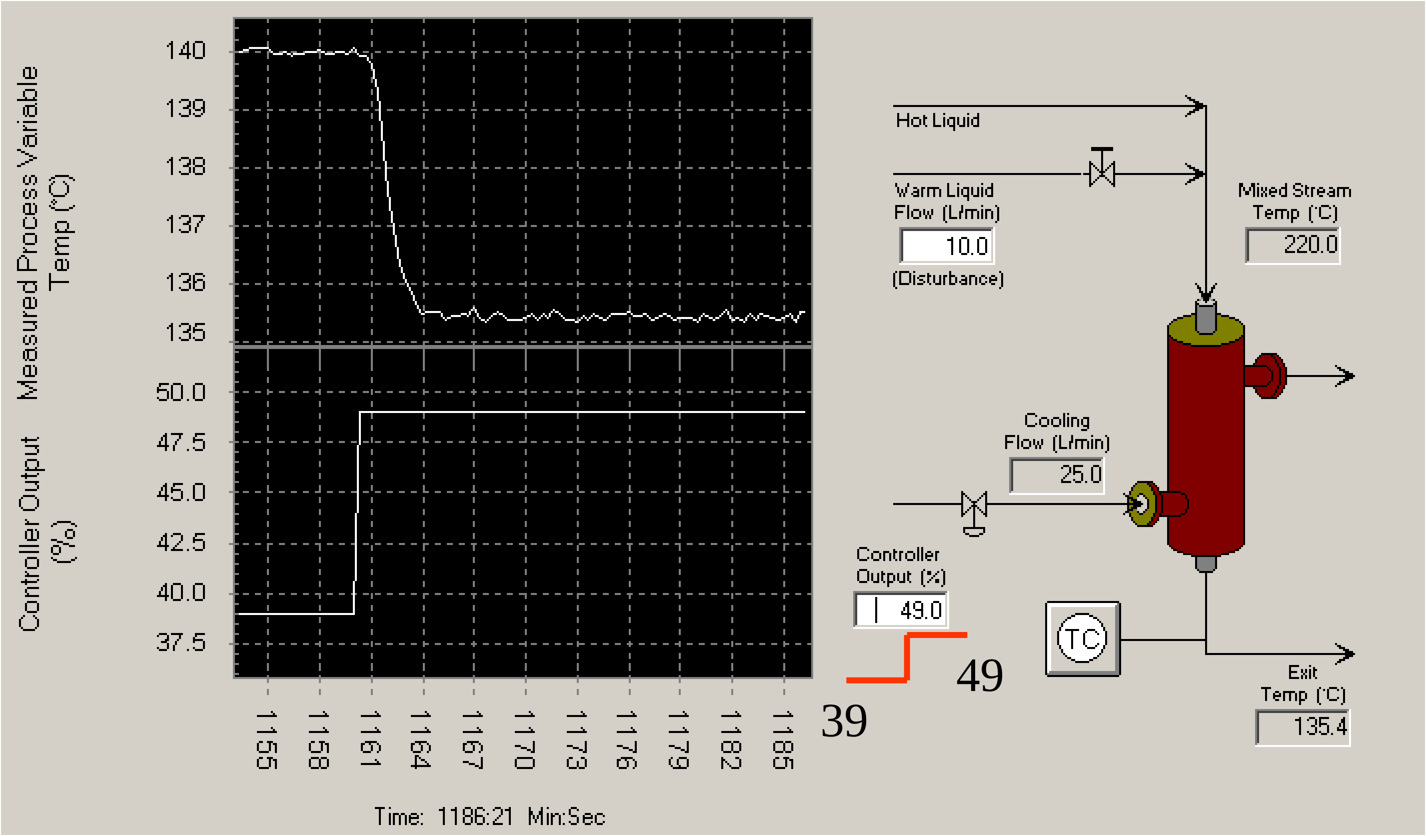
Cstation



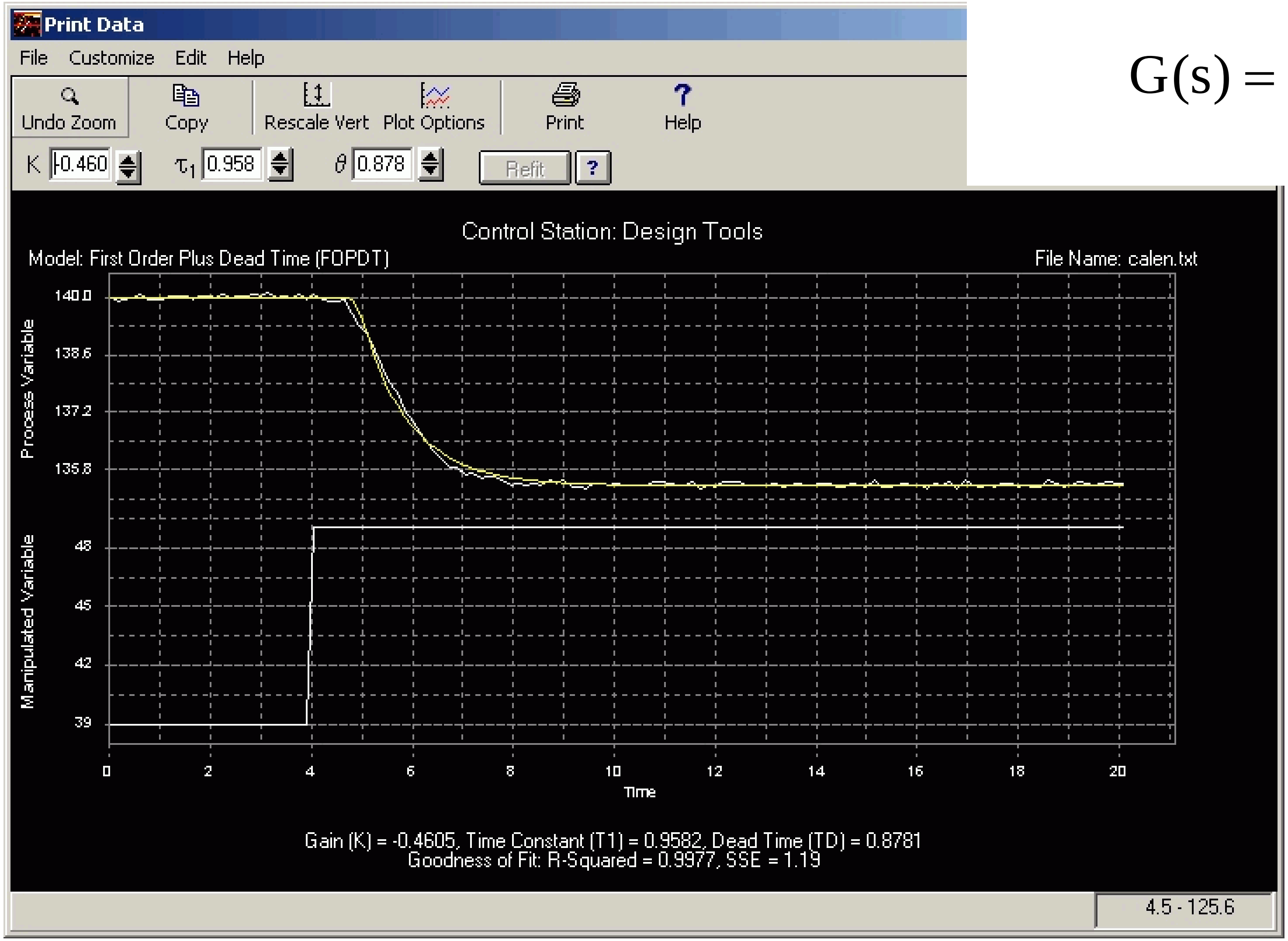
Gravity Drained Tank
File Run Tasks Help
Clock Rescale History Plot Save Edit File Pause Continue Navigate Help
5,0
4,5
4,0
cess Variable (m)
Inlet Flow (L/min)
17,8
Controller Output (%)
70,0
LC
Outlet Flow (L/min)
15,8
File Storage: OFF
Design Tools
File Tasks Help
Open Label Data Select Model Start Fitting Plot Implement Save Fit
First Order Plus Dead Time (Overdamped)
Nomenclature
Time Domain
$\tau_P \frac{dy(t)}{dt} + y(t) = K_P u(t - \theta_P)$
Laplace Domain
$\frac{Y(s)}{U(s)} = \frac{K_P e^{-\theta_P s}}{\tau_P s + 1}$

Heat exchanger (LS)

Open loop test



Heat exchanger (LS)



$$G(s) = \frac{ke^{-sd}}{(\tau s + 1)}$$

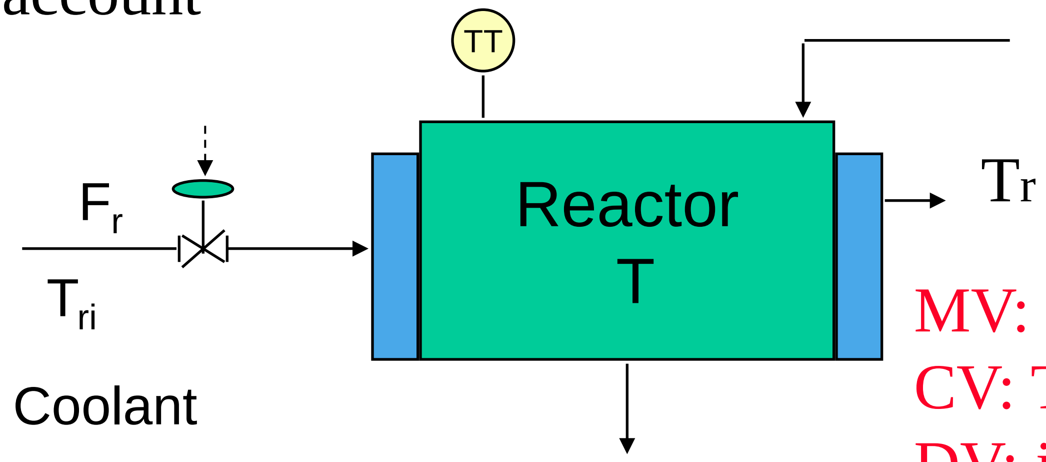
$$= \frac{-0.46e^{-0.87s}}{0.96s + 1}$$

Chemical reactor

Simplified model:

All variables related to the raw material, F, Ti, C_{ai}, are considered constant

The temperature is the only controlled variable taken into account

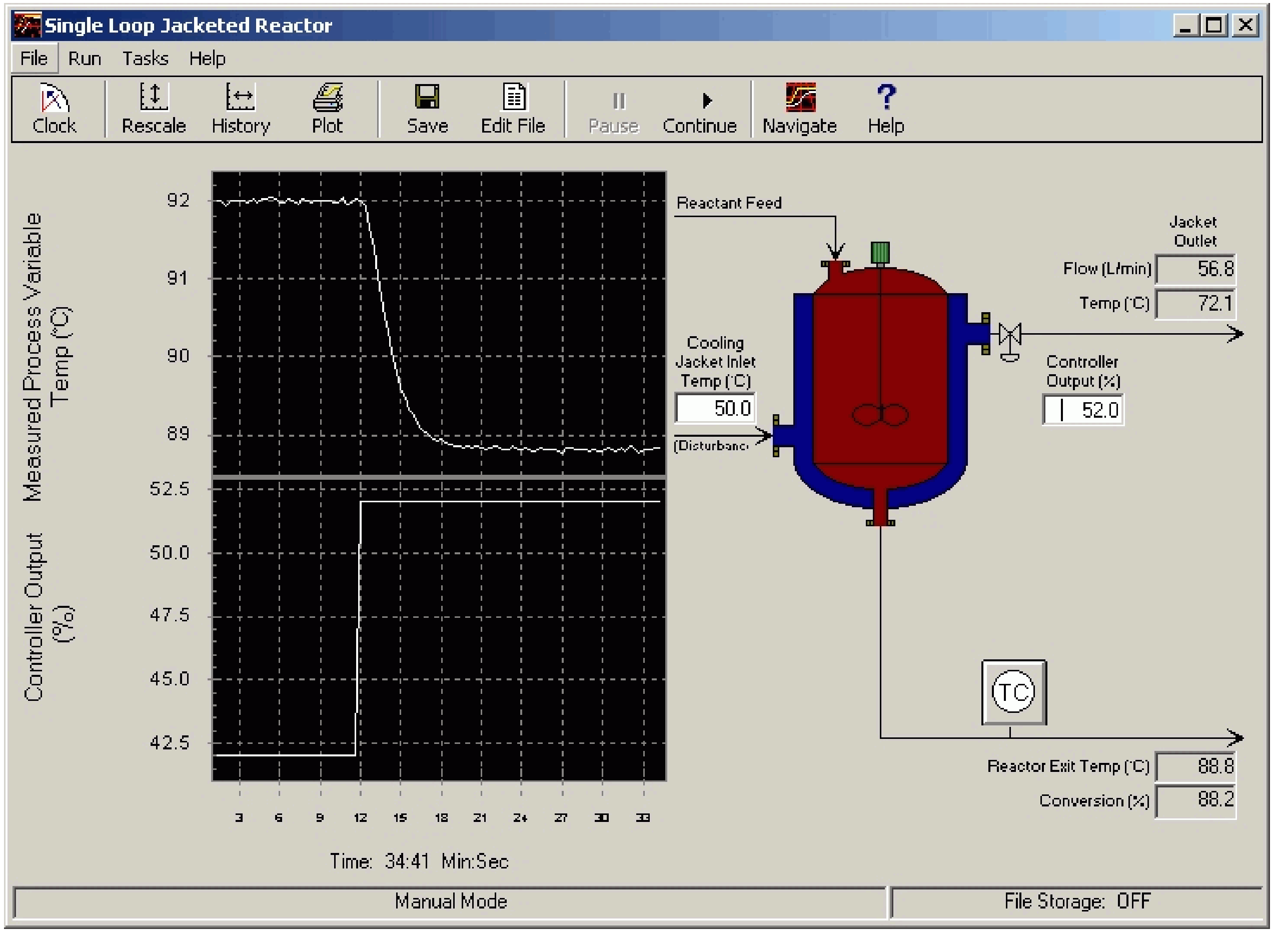


MV: coolant flow

CV: Temperature reactor

DV: input coolant temperature

Chemical Reactor - Temperature



Reduced model

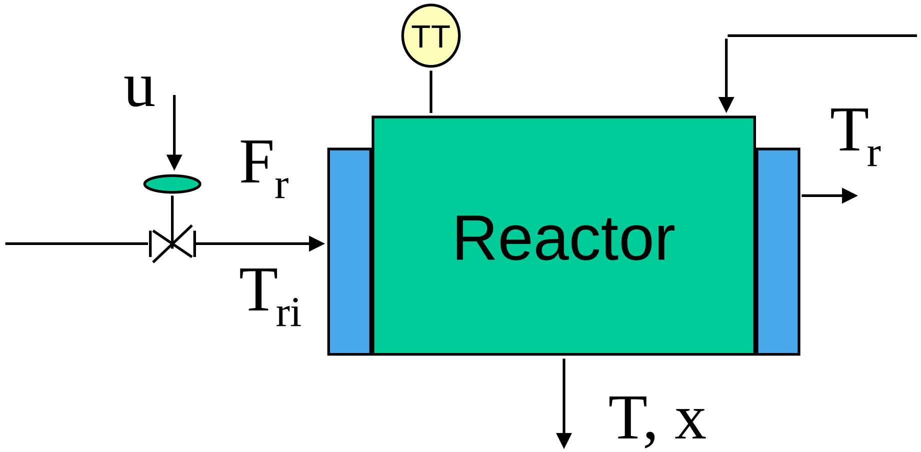
Conversion x $\qquad x = c_B/c_{Ai} \qquad c_A = c_{Ai}(1- x)$

$$V\frac{dc_A}{dt} = Fc_{Ai} - Fc_A - Vke^{-E/RT}c_A \quad \Rightarrow \quad \frac{dx}{dt} = -\frac{F}{V}x + ke^{-E/RT}(1-x)$$

$$V\rho c_e \frac{dT}{dt} = F\rho c_e T_i - F\rho c_e T + Vke^{-E/RT}c_A \Delta H - UA(T - T_r)$$

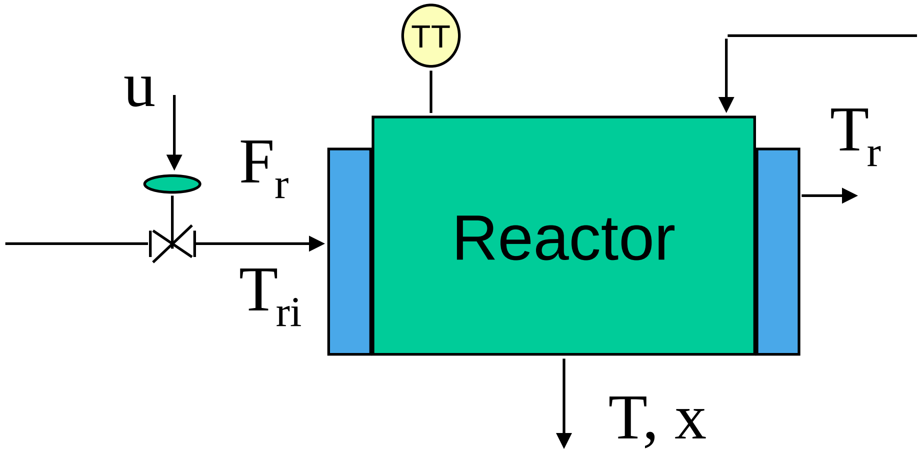
$$V_r\rho_r c_{er}\frac{dT_r}{dt} = F_r\rho_r c_{e_r} T_{ri} - F_r\rho_r c_{er} T_r + UA(T - T_r)$$

Parameter Estimation



In order to compute the model parameters (U, F_0, E,….) some measurements are required. Some parameters can be computed from data collected from CStation in steady state, but other parameters cannot be estimated from these data

Operating point



$T = 92\ ^oC \quad x = 0.902$
$T_r = 75.6\ ^oC$
$F_r = 47.8\ l/m$
$T_{ri} = 50\ ^oC \quad u = 42\ \%$

Other: $T = 88.6\ ^oC \quad x = 0.881 \quad T_r = 71.8\ ^oC$
$F_r = 30.\ l/m \quad T_{ri} = 30\ ^oC \quad u = 22.2\ \%$

Another: $T = 33.6\ ^oC \quad x = 0.102 \quad T_r = 32.2\ ^oC$
$F_r = 47.8\ l/m \quad T_{ri} = 30\ ^oC \quad u = 42\ \%$

Parameter estimation

$$0 = Fx - Vke^{-E/RT}(1-x)$$

$$0 = F(T_i - T) + \frac{Vke^{-E/RT}(1-x)c_{Ai}\Delta H}{\rho c_e} - \frac{UA}{\rho c_e}(T - T_r)$$

$$0 = F_r(T_{ri} - T_r) + \frac{UA}{\rho_r c_{e_r}}(T - T_r)$$

T = 92 ºC   x = 0.902
T_r = 75.6 ºC
F_r = 47.8 l/m
T_{ri} = 50 ºC   u = 42 %

$$0 = 0.902F - Vke^{-E/R(92+273.2)}(1-0.902) \Rightarrow \ln 0.902 + \ln\frac{F}{Vk} = -\frac{E}{R(92+273.2)} + \ln(1-0.902)$$

$$0 = F(T_i - 92) + \frac{Vke^{-E/R(92+273.2)}(1-0.902)c_{Ai}\Delta H}{\rho c_e} - \frac{UA}{\rho c_e}(92 - 75.6)$$

$$0 = 47.8(50 - 75.6) + \frac{UA}{\rho_r c_{e_r}}(92 - 75.6) \Rightarrow \frac{UA}{\rho_r c_{er}} = 74.5$$

Parameter estimation

$T = 92\ ^oC \quad x = 0.902$
$T_r = 75.6\ ^oC$
$F_r = 47.8\ l/m$
$T_{ri} = 50\ ^oC \quad u = 42\ \%$

$T = 24.5\ ^oC \quad x = 0.047$
$T_r = 21.9\ ^oC$
$F_r = 100\ l/m \quad T_{ri} = 20\ ^oC \quad u = 100\ \%$

$T = 88.8\ ^oC \quad x = 0.882$
$T_r = 72\ ^oC$
$F_r = 56.8\ l/m$
$T_{ri} = 50\ ^oC \quad u = 52\ \%$

$$\ln 0.902 + \ln\frac{F}{Vk} = -\frac{E}{R(92+273.2)} + \ln(1-0.902)$$

$$\ln 0.882 + \ln\frac{F}{Vk} = -\frac{E}{R(88.8+273.2)} + \ln(1-0.882)$$

$$0 = F(T_i - 92) + \frac{Vke^{-E/R(92+273.2)}(1-0.902)c_{Ai}\Delta H}{\rho c_e} - \frac{UA}{\rho c_e}(92-75.6)$$

$$0 = F(T_i - 88.8) + \frac{Vke^{-E/R(88.8+273.2)}(1-0.882)c_{Ai}\Delta H}{\rho c_e} - \frac{UA}{\rho c_e}(88.8-72)$$

$\Rightarrow$

$$\frac{E}{R} = 8598.9$$

$$\frac{F}{Vk} = 6.46e-012$$

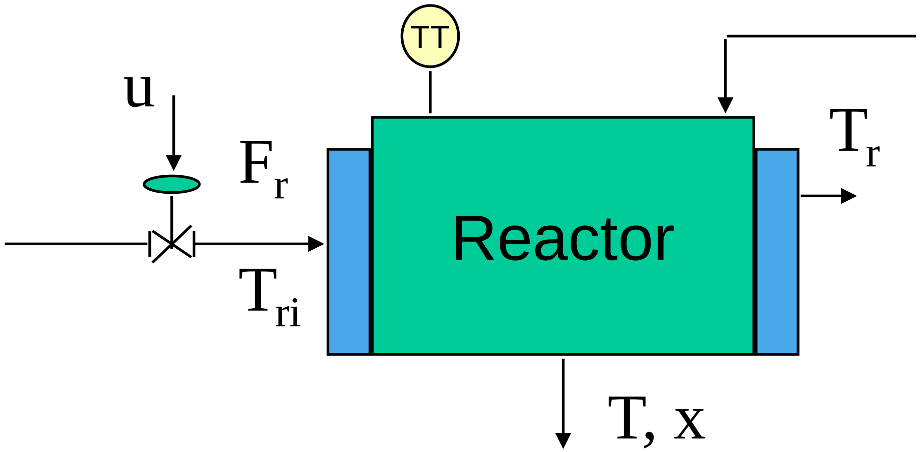
$$\frac{c_{Ai}\Delta H}{\rho c_e} = 114.783$$

$$\frac{UA}{\rho c_e Vk} = 1.460e-011$$

$$T_i = 25.54$$

Plus another one in the third point

Parameter estimation



Assuming:

$V = V_r = 68.8941$ l

$F = 34.4471$ l/min

$\rho c_e = 4180$ j/k l

$\rho_r c_{er} = 4000$ j/k l

$$\frac{c_{Ai}\Delta H}{\rho c_e} = 114.783 \qquad \frac{E}{R} = 8598.9$$

$$\frac{UA}{\rho c_e V k} = 1.460e-011 \qquad \frac{F}{Vk} = 6.46e-012$$

$$T_i = 25.54 \qquad \frac{UA}{\rho_r c_{er}} = 74.5$$

One can obtain:

$k = 7.7399e+010$

$c_{Ai}\Delta H = 479792.94$

$UA = 311410$

Reactor Matlab

Reduced model, linearization

$$\frac{dx}{dt} = -\frac{F}{V}x + ke^{-E/RT}(1-x)$$

$$\frac{d\Delta x}{dt} = -\left(\frac{F_0}{V} + ke^{-E/RT_0}\right)\Delta x + \frac{kE}{RT_0^2}e^{-E/RT_0}(1-x_0)\Delta T \quad \Rightarrow \quad \frac{d\Delta x}{dt} = a_{11}\Delta x + a_{12}\Delta T$$

$$V\rho c_e \frac{dT}{dt} = F\rho c_e T_i - F\rho c_e T + Vke^{-E/RT}c_{Ai}(1-x)\Delta H - UA(T - T_r)$$

$$\frac{d\Delta T}{dt} = \left(\frac{-ke^{-E/RT_0}c_{Ai}\Delta H}{\rho c_e}\right)\Delta x + \left(-\frac{F_0}{V} + \frac{kEe^{-E/RT_0}}{RT_0^2}\frac{c_{Ai}(1-x_0)\Delta H}{\rho c_e} - \frac{UA}{V\rho c_e}\right)\Delta T +$$

$$+\left(\frac{UA}{V\rho c_e}\right)\Delta T_r \quad \Rightarrow \quad \frac{d\Delta T}{dt} = a_{21}\Delta x + a_{22}\Delta T + a_{23}\Delta T_r$$

Reduced model, linearization

$$V_r\rho_r c_{er}\frac{dT_r}{dt} = F_r\rho_r c_{e_r} T_{ri} - F_r\rho_r c_{er} T_r + UA(T - T_r)$$

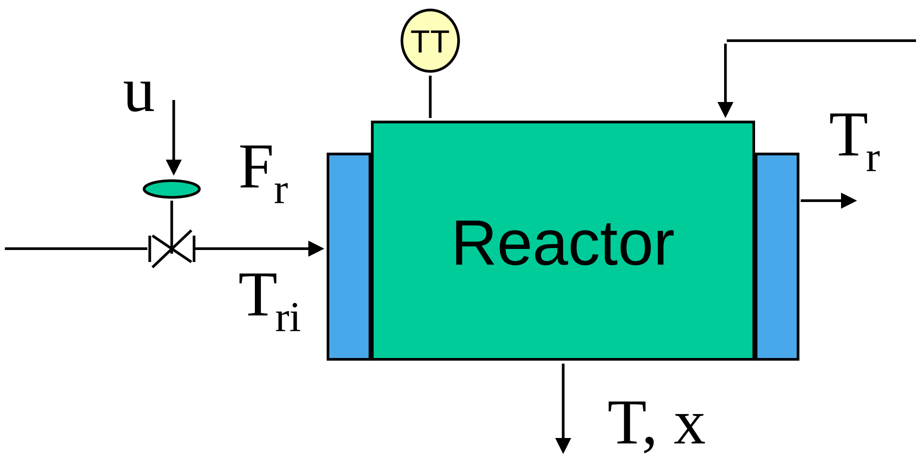
$$\frac{d\Delta T_r}{dt} = \left(\frac{UA}{V_r\rho_r c_{er}}\right)\Delta T - \left(\frac{UA}{V_r\rho_r c_{er}} + \frac{F_{r0}}{V_r}\right)\Delta T_r + \left(\frac{T_{ri0} - T_{r0}}{V_r}\right)\Delta F_r + \left(\frac{F_{r0}}{V_r}\right)\Delta T_{ri} \quad \Rightarrow$$

$$\frac{d\Delta T_r}{dt} = a_{32}\Delta T + a_{33}\Delta T_r + b_{31}\Delta F_r + b_{32}\Delta T_{ri}$$

$$\begin{bmatrix}\Delta\dot{x}\\ \Delta\dot{T}\\ \Delta\dot{T}_r\end{bmatrix} = \begin{pmatrix} a_{11} & a_{12} & 0\\ a_{21} & a_{22} & a_{23}\\ 0 & a_{32} & a_{33}\end{pmatrix}\begin{bmatrix}\Delta x\\ \Delta T\\ \Delta T_r\end{bmatrix} + \begin{pmatrix} 0 & 0\\ 0 & 0\\ b_{32} & b_{32}\end{pmatrix}\begin{bmatrix}\Delta F_r\\ \Delta T_{ri}\end{bmatrix}$$

$$\Delta T = \begin{pmatrix}0 & 1 & 0\end{pmatrix}\begin{bmatrix}\Delta x\\ \Delta T\\ \Delta T_r\end{bmatrix} + \begin{pmatrix}0 & 0\end{pmatrix}\begin{bmatrix}\Delta F_r\\ \Delta T_{ri}\end{bmatrix}$$

Linearized model



In the operating point:

T = 92 ºC   x = 0.902
T_r = 75.6 ºC
F_r = 47.8 l/m
T_{ri} = 50 ºC   u = 42 %

$$\frac{d\Delta x}{dt} = -\left(\frac{F_0}{V} + k e^{-E/RT_0}\right)\Delta x + \frac{kE}{RT_0^2} e^{-E/RT_0}(1 - x_0)\Delta T$$

Assigning values to:

$$\frac{d\Delta T}{dt} = \left(\frac{-k e^{-E/RT_0} c_{Ai}\Delta H}{\rho c_e}\right)\Delta x + \left(-\frac{F_0}{V} + \frac{kE e^{-E/RT_0}}{RT_0^2}\frac{c_{Ai}(1 - x_0)\Delta H}{\rho c_e} - \frac{UA}{V\rho c_e}\right)\Delta T + \left(\frac{UA}{V\rho c_e}\right)\Delta T_r$$

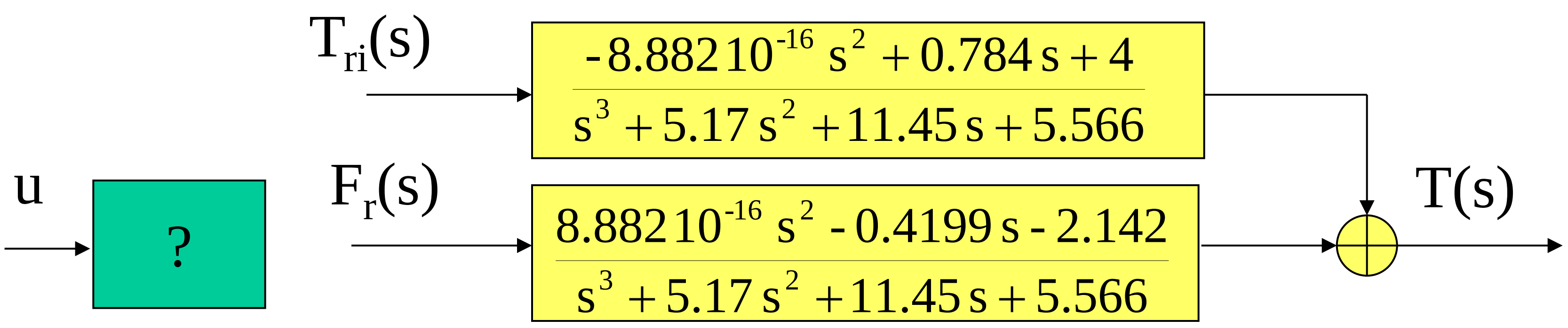
$$\frac{d\Delta T_r}{dt} = \left(\frac{UA}{V_r \rho_r c_{er}}\right)\Delta T - \left(\frac{UA}{V_r \rho_r c_{er}} + \frac{F_{r0}}{V_r}\right)\Delta T_r + \left(\frac{T_{ri0} - T_{r0}}{V_r}\right)\Delta F_r + \left(\frac{F_{r0}}{V_r}\right)\Delta T_{ri}$$

State space / Block diagram

$$\begin{bmatrix} \Delta\dot{x} \\ \Delta\dot{T} \\ \Delta\dot{T}_r \end{bmatrix} = \begin{pmatrix} -5.1 & 0.029 & 0 \\ -528.2 & 1.707 & 1.13 \\ 0 & 1.081 & -1.77 \end{pmatrix} \begin{bmatrix} \Delta x \\ \Delta T \\ \Delta T_r \end{bmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -0.37 & 0.694 \end{pmatrix} \begin{bmatrix} \Delta F_r \\ \Delta T_{ri} \end{bmatrix}$$

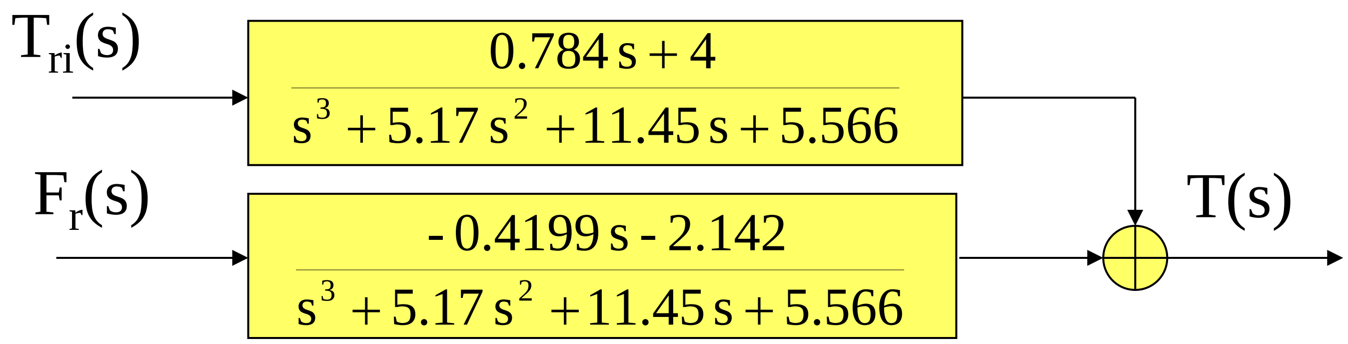
$$\Delta T = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{bmatrix} \Delta x \\ \Delta T \\ \Delta T_r \end{bmatrix} + \begin{pmatrix} 0 & 0 \end{pmatrix} \begin{bmatrix} \Delta F_r \\ \Delta T_{ri} \end{bmatrix}$$

$$G(s) = C[sI - A]^{-1} B$$



$F_r = 0.9u + 10$

Reactor Model in s

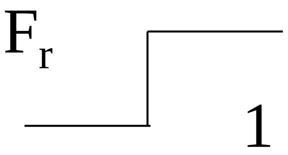


Roots (denominator)
-2.2571 + 1.8435i
-2.2571 - 1.8435i
-0.6554

Zeros	Gain
-5.1 (T_{ri})	0.718
- 5.1 (F_r)	- 0.385

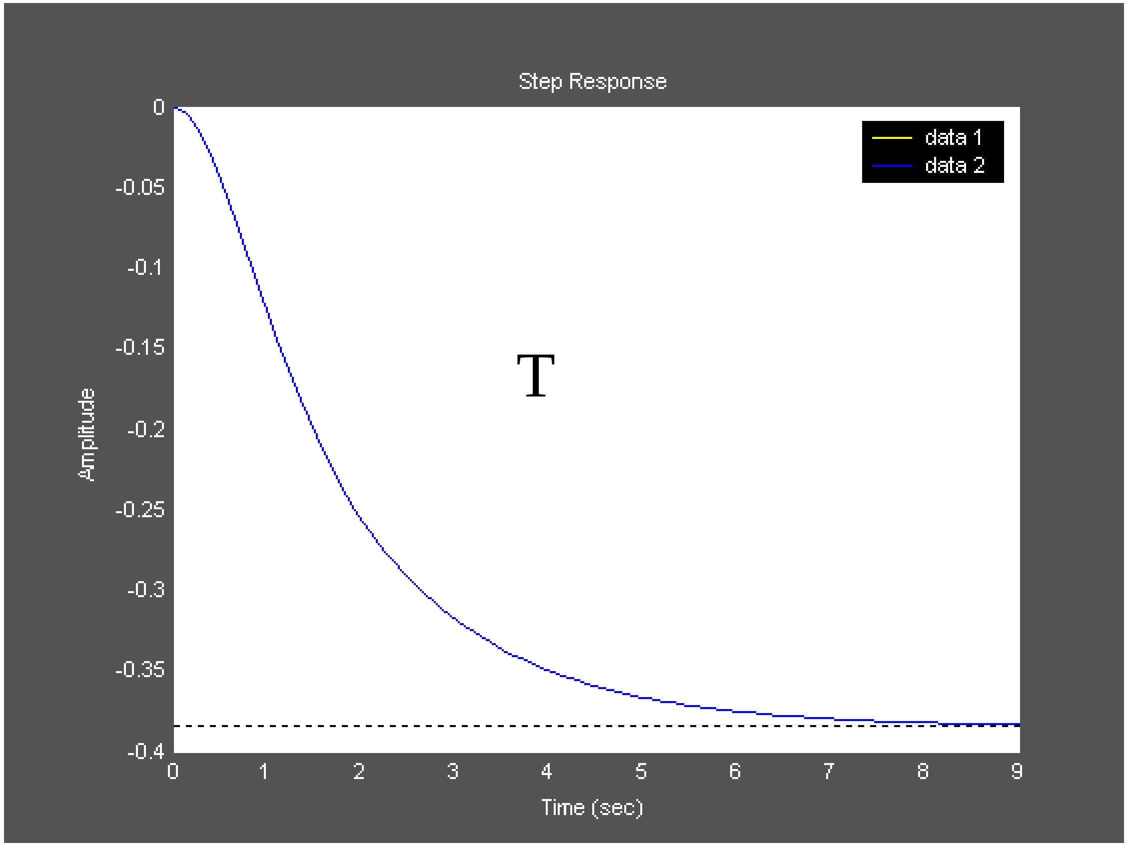
Stable operating point

Step response

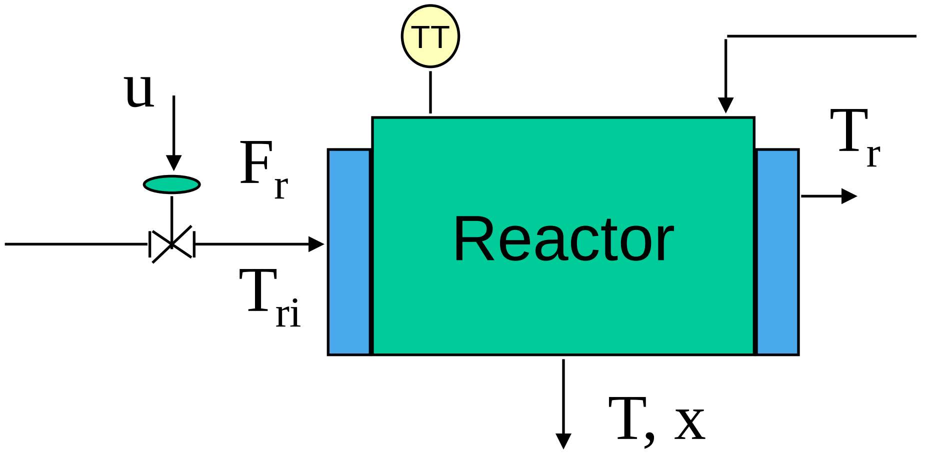


roots(d2)
-2.2571 + 1.8435i
-2.2571 - 1.8435i
-0.6554

Dominant pole



Other operating point



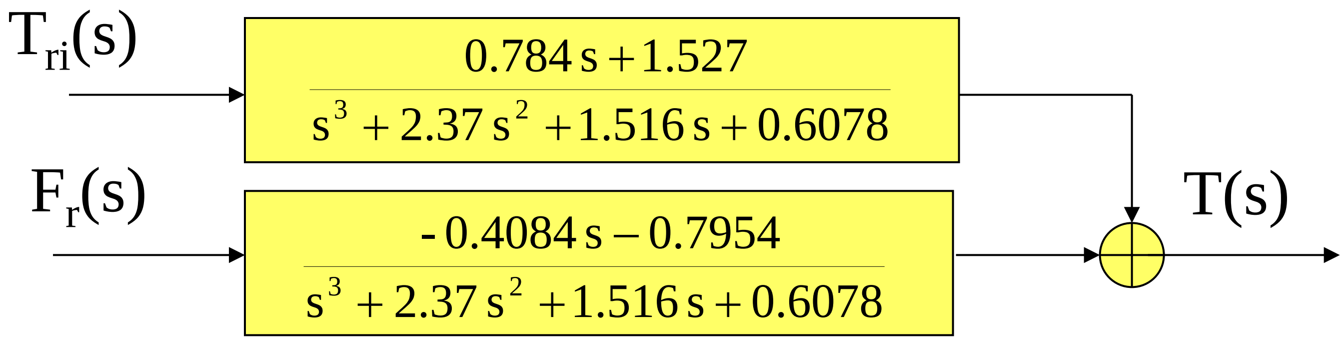
Operating point

$T = 74.9\ ^oC \quad x = 0.747$

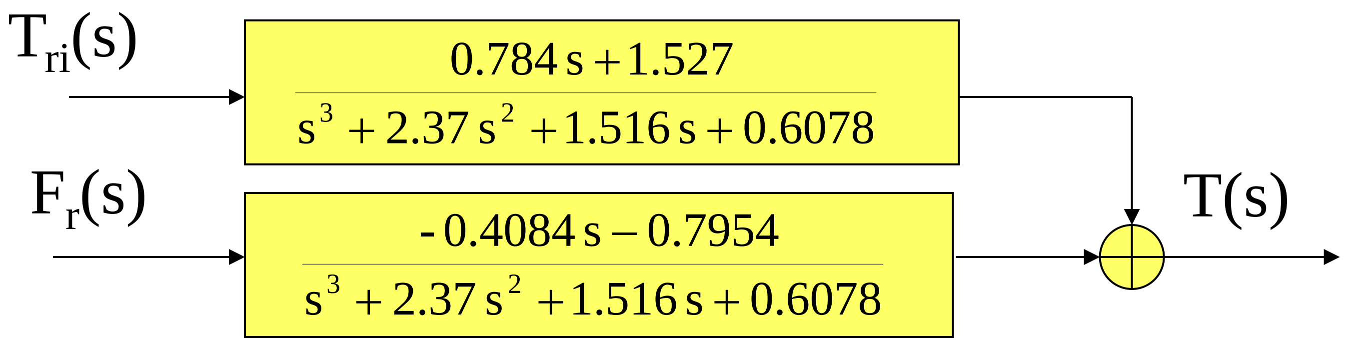
$T_r = 58.9\ ^oC$

$F_r = 47.8\ l/m$

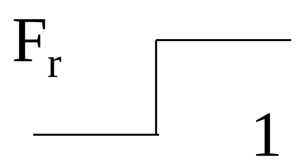
$T_{ri} = 34\ ^oC \quad u = 42\ \%$

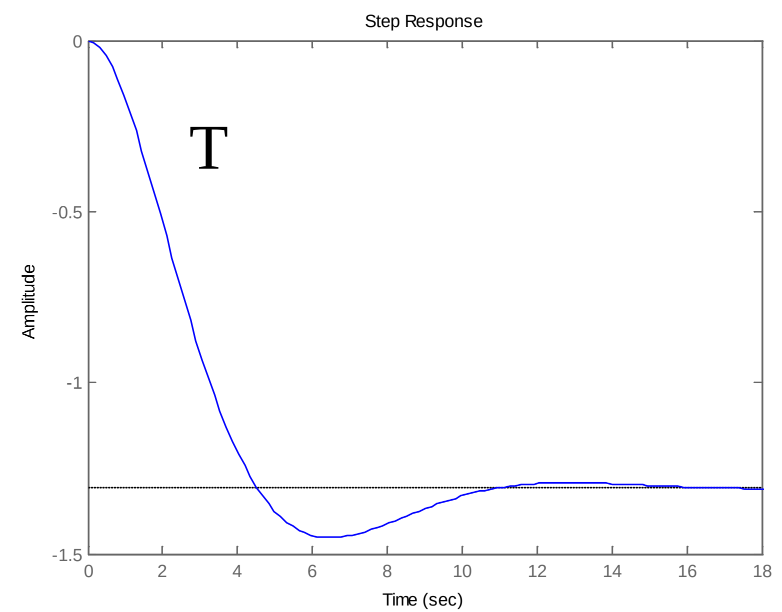


Other operating point

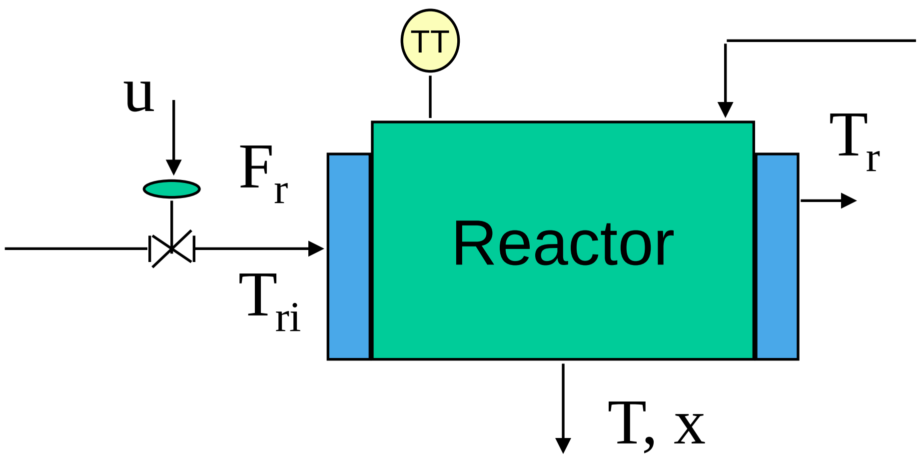


Poles:
- -1.6834
- -0.3432 + 0.4933i
- -0.3432 - 0.4933i





An unstable operating point



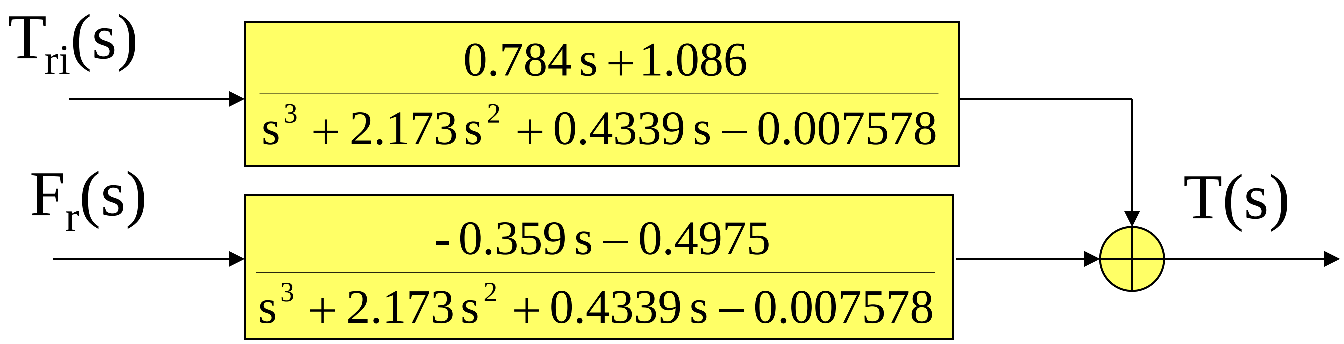
Operating point:

$T = 68.1\ ^{o}C \quad x = 0.651$

$T_r = 54.6\ ^{o}C$

$F_r = 47.8\ l/m$

$T_{ri} = 32.7\ ^{o}C \quad u = 42\ \%$



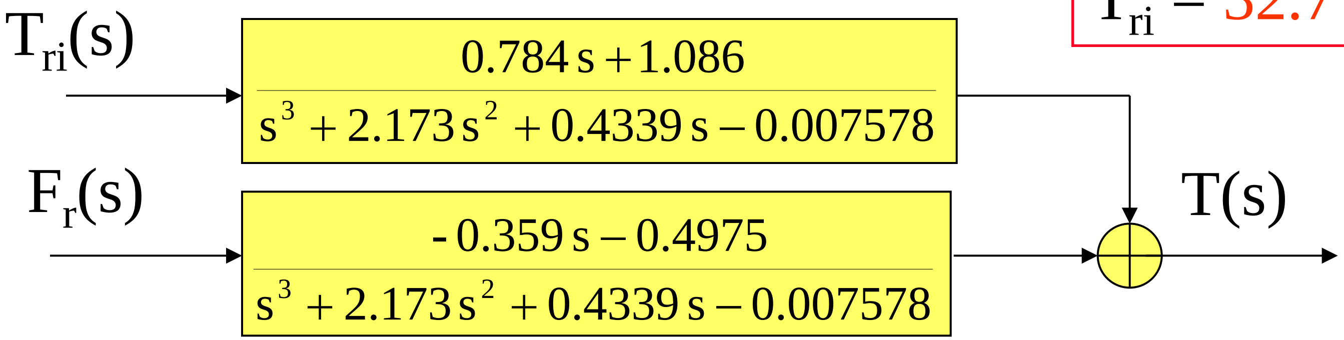
An unstable operating point

$T = 68.1\ ^{o}C \quad x = 0.651$

$T_r = 54.6\ ^{o}C$

$F_r = 47.8\ l/m$

$T_{ri} = 32.7\ ^{o}C \quad u = 42\ \%$

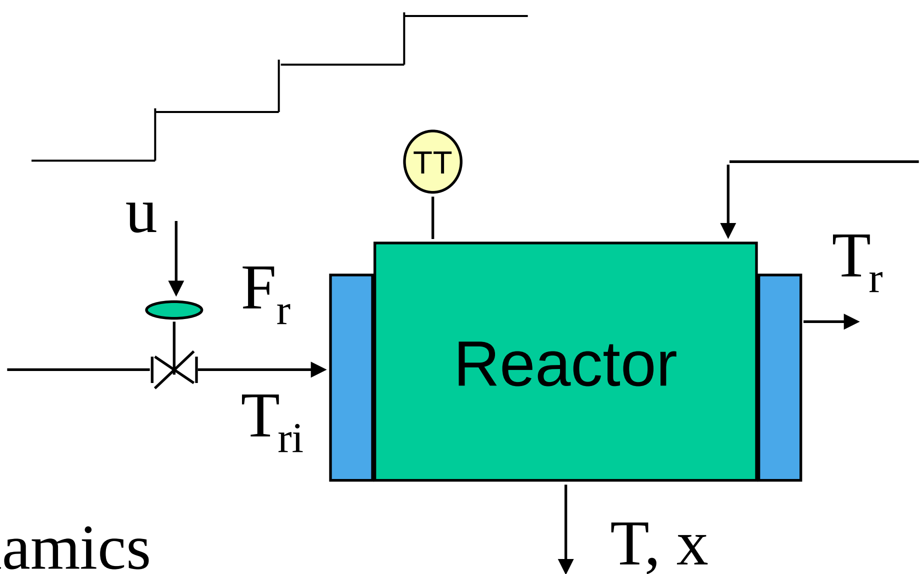


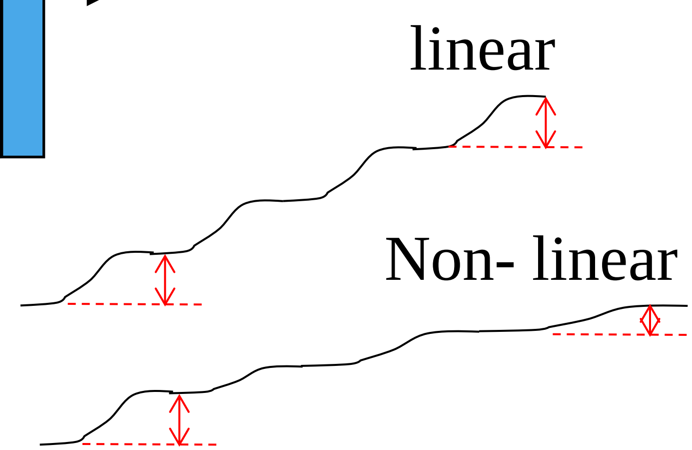
Poles: -1.9487

-0.2408

0.0161

How can we distinguish if a process is linear or non-linear?





If the dynamics change with the operating point, the process is non-linear