Process Modelling (2)

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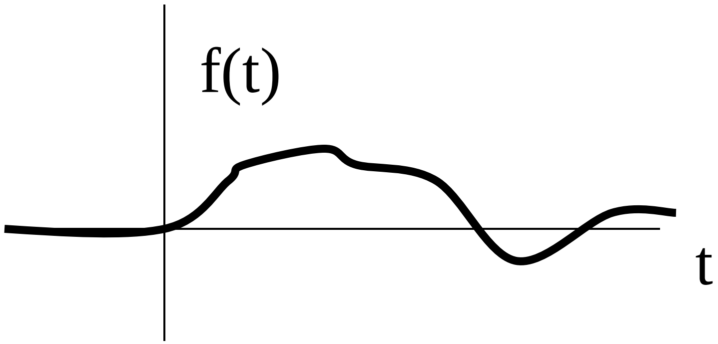
Outline

- Laplace transform
- Transfer functions
- Examples
- Block diagrams
- Poles and ceros
- Delays

The Laplace Transform

f(t) time function

$f(t) = 0$ for $t < 0$



$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty f(t) e^{-st} dt$$

$s = \sigma + j\omega$ Laplace complex variable

if $f(t) = g(t)$

$\mathcal{L}[f(t)] = \mathcal{L}[g(t)]$

$F(s) = G(s)$

Change of variable $t \Rightarrow s$

The Laplace Transform

$$\text{if} \quad f(t) = g(t)$$

$$L[f(t)] = L[g(t)]$$

$$F(s) = G(s)$$

Change of variable $t \Rightarrow s$

The problem is solved in the s domain $X(s)$

Then the solution is converted back to the time domain

$$x(t) = L^{-1}[X(s)] = \int_{-j\infty}^{j\infty} X(s)e^{st}ds$$

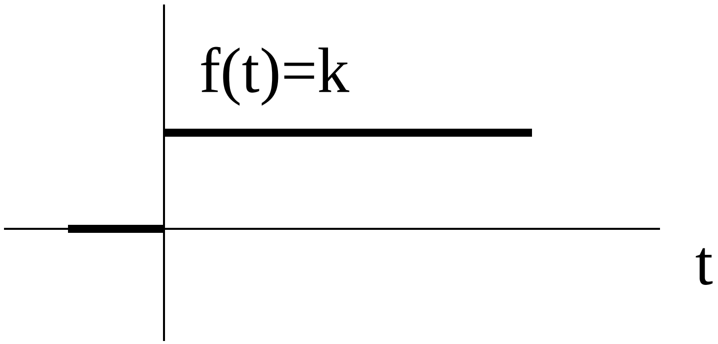
Change of variable $s \Rightarrow t$

Example

f(t) step function

f(t) = 0 for t < 0

f(t) = k for t >= 0



$$L\left[f(t)\right] = F(s) = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty ke^{-st}dt = -k\left.\frac{e^{-st}}{s}\right|_0^\infty = \frac{k}{s}$$

There are tables of Laplace transforms for the most common functions

Laplace Transforms table

	$f(t)$	$F(s)$
1	Impulso unitario $\delta(t)$	1
2	Escalón unitario $1(t)$	$\frac{1}{s}$
3	t	$\frac{1}{s^2}$
4	$\frac{t^{n-1}}{(n-1)!} \quad (n = 1, 2, 3, \ldots)$	$\frac{1}{s^n}$
5	e^{-at}	$\frac{1}{s+a}$
6	te^{-at}	$\frac{1}{(s+a)^2}$
7	$\frac{1}{(n-1)!} t^{n-1} e^{-at} \quad (n = 1, 2, 3, \ldots)$	$\frac{1}{(s+a)^n}$
8	$\frac{1}{b-a}(e^{-at} - e^{-bt})$	$\frac{1}{(s+a)(s+b)}$

Laplace Transforms table

9	$\frac{1}{ab}\left[1+\frac{1}{a-b}\left(be^{-at}-ae^{-bt}\right)\right]$	$\frac{1}{s(s+a)(s+b)}$
10	$\text{sen}\ \omega t$	$\frac{\omega}{s^2+\omega^2}$
11	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
12	$e^{-at}\ \text{sen}\ \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
13	$e^{-at}\cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$
14	$\frac{\omega_n}{\sqrt{1-\zeta^2}}\,e^{-\zeta\omega_n t}\ \text{sen}\ \omega_n\sqrt{1-\zeta^2}\,t$	$\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$
15	$-\frac{1}{\sqrt{1-\zeta^2}}\,e^{-\zeta\omega_n t}\ \text{sen}\left(\omega_n\sqrt{1-\zeta^2}\,t-\phi\right)$ $\phi=\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$	$\frac{s}{s^2+2\zeta\omega_n s+\omega_n^2}$
16	$1-\frac{1}{\sqrt{1-\zeta^2}}\,e^{-\zeta\omega_n t}\ \text{sen}\left(\omega_n\sqrt{1-\zeta^2}\,t+\phi\right)$ $\phi=\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$	$\frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}$

Properties of the Laplace Trans.

$$L[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}dt$$

$$L[af(t) + bg(t)] = aF(s) + bG(s)$$

$$L\left[\frac{df(t)}{dt}\right] = sF(s) - f(0) \qquad L\left[\frac{d^2f(t)}{dt^2}\right] = s^2F(s) - s\frac{df(0)}{dt} - f(0)$$

$$L[f(t-d)] = e^{-sd}F(s)$$

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

$$L\left[\int_0^\infty f(\tau)g(t-\tau)d\tau\right] = F(s)G(s)$$

Inverse Transform

$$f(t) = L^{-1}[F(s)] = \int_{-j\infty}^{j\infty} F(s)e^{st}ds$$

Properties I

$$L\left[f(t)\right] = F(s) = \int_0^\infty f(t)e^{-st}dt$$

$$L\left[af(t) + bg(t)\right] = aF(s) + bG(s)$$

$$L\left[af(t) + bg(t)\right] = \int_0^\infty \left[af(t) + bg(t)\right]e^{-st}dt = a\int_0^\infty f(t)e^{-st}dt + b\int_0^\infty g(t)e^{-st}dt = aF(s) + bG(s)$$

$$L\left[\frac{df(t)}{dt}\right] = sF(s) - f(0) \qquad L\left[\frac{df(t)}{dt}\right] = \int_0^\infty \frac{df(t)}{dt}e^{-st}dt$$

$$\int u\,dv = uv - \int v\,du \qquad dv = \frac{df(t)}{dt}dt \quad u = e^{-st} \quad \Rightarrow v = f(t) \quad du = -se^{-st}dt$$

$$L\left[\frac{df(t)}{dt}\right] = \int_0^\infty \frac{df(t)}{dt}e^{-st}dt = \left[e^{-st}f(t)\right]_0^\infty + \int_0^\infty f(t)se^{-st}dt = -f(0) + sF(s)$$

Properties

$$\frac{d\int_0^t f(\tau)d\tau}{dt} = f(t) \qquad L\left[\frac{d\int_0^t f(\tau)d\tau}{dt}\right] = L[f(t)] = F(s)$$

$$L\left[\frac{d\int_0^t f(\tau)d\tau}{dt}\right] = sL\left[\int_0^t f(\tau)d\tau\right] - \int_0^0 f(\tau)d\tau = sL\left[\int_0^t f(\tau)d\tau\right]$$

$$L\left[\int_0^t f(\tau)d\tau\right] = \frac{1}{s}F(s)$$

Properties II

$$L\left[f(t-d)\right] = e^{-sd}F(s)$$

$$L\left[f(t-d)\right] = \int_0^\infty f(t-d)e^{-st}dt \qquad t-d=\tau \quad t=0 \Rightarrow \tau=-d; \quad t=\infty \Rightarrow \tau=\infty$$

$$\int_0^\infty f(t-d)e^{-st}dt = \int_{-d}^\infty f(\tau)e^{-s(\tau+d)}d\tau = \int_0^\infty f(\tau)e^{-sd}e^{-s\tau}d\tau = e^{-sd}\int_0^\infty f(\tau)e^{-s\tau}d\tau = e^{-sd}F(s)$$

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) \qquad sF(s) = \int_0^\infty \frac{d\,f(t)}{d\,t}e^{-st}dt + f(0)$$

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0}\int_0^\infty \frac{d\,f(t)}{d\,t}e^{-st}dt + f(0) = \int_0^\infty \frac{d\,f(t)}{d\,t}dt + f(0) =$$

$$= f(t)\Big|_0^\infty + f(0) = f(\infty) - f(0) + f(0) = f(\infty)$$

Properties III

$$L\left[\int_0^\infty f(\tau)g(t-\tau)d\tau\right] = F(s)G(s)$$

$$L\left[\int_0^\infty f(\tau)g(t-\tau)d\tau\right] = \int_0^\infty\left[\int_0^\infty f(\tau)g(t-\tau)d\tau\right]e^{-st}dt$$

$$t-\tau=\alpha \qquad t=0 \Rightarrow \alpha=-\tau; \quad t=\infty \Rightarrow \alpha=\infty$$

$$\int_0^\infty\left[\int_0^\infty f(\tau)g(t-\tau)d\tau\right]e^{-st}dt = \int_0^\infty\left[\int_0^\infty f(\tau)g(t-\tau)e^{-st}d\tau\right]dt = \int_{-\tau}^\infty\left[\int_0^\infty f(\tau)g(\alpha)e^{-s(\alpha+\tau)}d\tau\right]d\alpha =$$

$$= \int_{-\tau}^\infty\left[\int_0^\infty f(\tau)e^{-s\tau}d\tau\right]g(\alpha)e^{-s\alpha}d\alpha = \left[\int_0^\infty f(\tau)e^{-s\tau}d\tau\right]\int_{-\tau}^\infty g(\alpha)e^{-s\alpha}d\alpha =$$

$$= \left[\int_0^\infty f(\tau)e^{-s\tau}d\tau\right]\int_0^\infty g(\alpha)e^{-s\alpha}d\alpha = F(s)G(s)$$

Solving linear ODEs

Example:

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = \frac{du}{dt} - 0.5u \qquad y(0) = 0; \quad \frac{dy(0)}{dt} = 0; \quad u(t) = e^{-2t} \;\; \text{for } t \geq 0$$

$$L\left[\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y\right] = L\left[\frac{du}{dt} - 0.5u\right]$$

$$s^2Y(s) + 2sY(s) + Y(s) = sU(s) - 0.5U(s) \qquad Y(s)\left(s^2 + 2s + 1\right) = (s - 0.5)U(s)$$

$$Y(s) = \frac{s - 0.5}{s^2 + 2s + 1}U(s) \qquad U(s) = \frac{1}{s+2} \qquad Y(s) = \frac{s - 0.5}{s^2 + 2s + 1}\frac{1}{s+2}$$

$$y(t) = L^{-1}[Y(s)] = L^{-1}\left[\frac{s - 0.5}{s^2 + 2s + 1}\frac{1}{s+2}\right] = \ldots\ldots$$

time domain $\Rightarrow$ s domain $\Rightarrow$ time domain

Simple fraction decomposition

$$y(t) = L^{-1}[Y(s)] = L^{-1}\left[\frac{s-0.5}{s^2+2s+1}\frac{1}{s+2}\right] = L^{-1}\left[\frac{s-0.5}{(s+1)^2}\frac{1}{s+2}\right]$$

$$\frac{s-0.5}{(s+1)^2}\frac{1}{s+2} = \frac{a}{s+2} + \frac{b}{s+1} + \frac{c}{(s+1)^2}$$

Partial fraction expansion

$$\frac{s-0.5}{(s+1)^2}\frac{1}{s+2} = \frac{a(s+1)^2}{(s+1)^2(s+2)} + \frac{b(s+1)(s+2)}{(s+1)^2(s+2)} + \frac{c(s+2)}{(s+1)^2(s+2)}$$

$$s = -1 \quad \Rightarrow \quad -1.5 = c$$

$$s = -2 \quad \Rightarrow \quad -2.5 = a$$

$$s = 0 \quad \Rightarrow \quad -0.5 = a + 2b + 2c = -5.5 + 2b \quad \Rightarrow \quad b = 2.5$$

$$y(t) = L^{-1}\left[\frac{-2.5}{s+2} + \frac{2.5}{s+1} + \frac{-1.5}{(s+1)^2}\right] = -2.5e^{-2t} + 2.5e^{-t} - 1.5te^{-t}$$

Transfer Function

$$y(t) = \int_0^t g(\sigma)u(t-\sigma)d\sigma$$

Taking Laplace transforms on both sides:

$$Y(s) = L[y(t)] = L\left[\int_0^t g(\sigma)u(t-\sigma)d\sigma\right] =$$

$$= L[g(t)]L[u(t)] = G(s)U(s)$$

$$Y(s) = G(s)U(s) \qquad G(s) = \frac{Y(s)}{U(s)}$$

s complex variable

Transfer Function

$$\frac{dx}{dt} = Ax + Bu$$

$$y = Cx$$

Taking Laplace transforms with zero initial conditions:

$$sX(s) = AX(s) + BU(s) \qquad [sI - A]X(s) = BU(s)$$

$$Y(s) = CX(s)$$

$$X(s) = [sI - A]^{-1}BU(s) \qquad Y(s) = C[sI - A]^{-1}BU(s)$$

$$Y(s) = G(s)U(s) \qquad G(s) = C[sI - A]^{-1}B = L[g(t)]$$

Transfer Function

$$G(s) = C[sI - A]^{-1}B$$

It contains only rational operations: + - * /

G(s) is a rational function in the s variable

$$G(s) = C[sI - A]^{-1}B = \frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0}$$

$$G(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0} = \frac{N(s)}{D(s)}$$

Mathematical descriptions of linearized models

State space

$$\frac{dx}{dt} = Ax + Bu$$

$$y = Cx$$

$$y(t) = \int_0^t g(\sigma)u(t-\sigma)d\sigma$$

Impulse response

$$G(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0} = \frac{N(s)}{D(s)}$$

Transfer function

Realization of a Transfer function

There are many equivalent realizations. Minimal realization

State space

$$\frac{dx}{dt} = Ax + Bu$$

$$y = Cx$$

$$y(t) = \int_0^t g(\sigma)u(t-\sigma)d\sigma$$

Impulse response

$$G(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0} = \frac{N(s)}{D(s)}$$

Transfer function

Matlab functions ss2tf, tf2ss, etc.

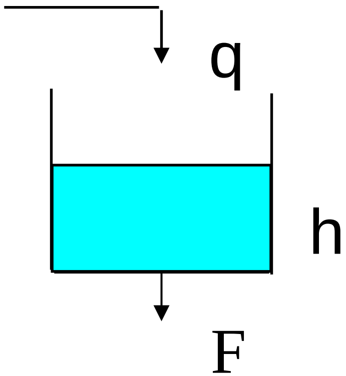
Transfer matrix

If the process has several inputs and outputs (MIMO) G(s) is a matrix which elements are transfer functions

$$G(s) = C[sI - A]^{-1}B$$

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \\ Y_3(s) \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \\ G_{31}(s) & G_{32}(s) \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix}$$

Tank. TF model

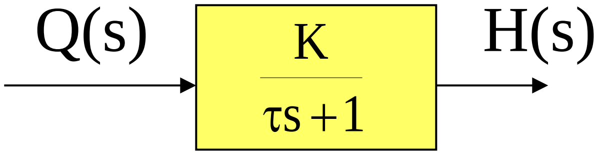


$$\tau \frac{d\Delta h}{dt} + \Delta h = K\Delta q$$

$$\tau = \frac{A2\sqrt{h_0}}{k} \qquad K = \frac{2\sqrt{h_0}}{k}$$

Taking Laplace transforms with zero IC:

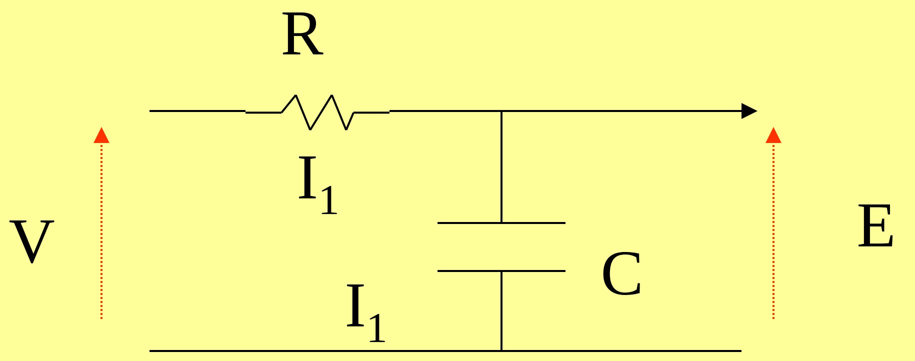
$$\mathcal{L}\left[\tau \frac{d\Delta h}{dt} + \Delta h\right] = \mathcal{L}[K\Delta q]$$



$$\tau s H(s) + H(s) = KQ(s) \qquad H(s)(\tau s + 1) = KQ(s)$$

$$H(s) = \frac{K}{\tau s + 1} Q(s) \qquad H(s) = G(s)Q(s) \qquad G(s) = \frac{K}{\tau s + 1}$$

RC Circuit. TF Model



$$V = I_1 R + \frac{1}{C}\int I_1 dt$$

$$E = \frac{1}{C}\int I_1 dt$$

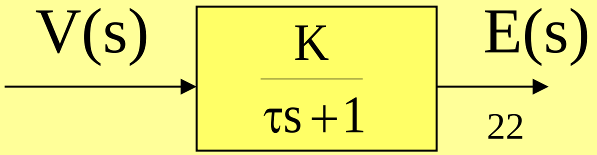
Taking Laplace transforms with zero initial conditions:

$$V(s) = I_1(s)R + \frac{1}{Cs} I_1(s)$$

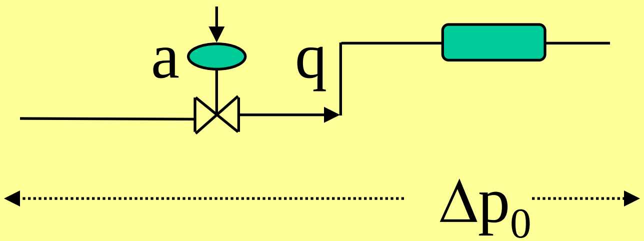
$$E(s) = \frac{1}{Cs} I_1(s)$$

$$V(s) = I_1(s)R + \frac{1}{Cs} I_1(s) = \frac{(RCs+1)}{Cs} I_1(s)$$

$$E(s) = \frac{1}{Cs} I_1(s) = \frac{1}{RCs+1} V(s)$$



Flow. TF Model



$$\tau \frac{d\Delta q}{dt} + \Delta q = K_1 \Delta(\Delta p_0) + K_2 \Delta a$$

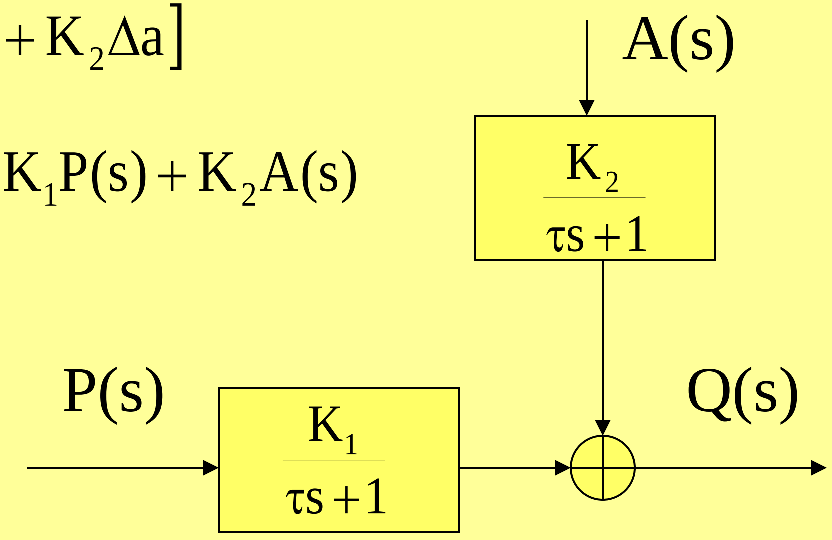
Taking Laplace transforms with zero IC:

$$L\left[\tau \frac{d\Delta q}{dt} + \Delta q\right] = L\left[K_1 \Delta(\Delta p_0) + K_2 \Delta a\right]$$

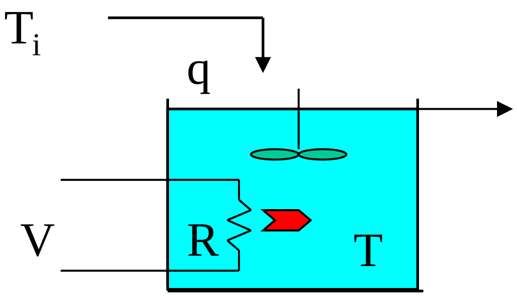
$$\tau s Q(s) + Q(s) = Q(s)(\tau s + 1) = K_1 P(s) + K_2 A(s)$$

$$Q(s) = \frac{K_1}{\tau s + 1} P(s) + \frac{K_2}{\tau s + 1} A(s)$$

$$Q(s) = \begin{bmatrix} \frac{K_1}{\tau s + 1} & \frac{K_2}{\tau s + 1} \end{bmatrix} \begin{bmatrix} P(s) \\ A(s) \end{bmatrix}$$



Temperature. TF Model



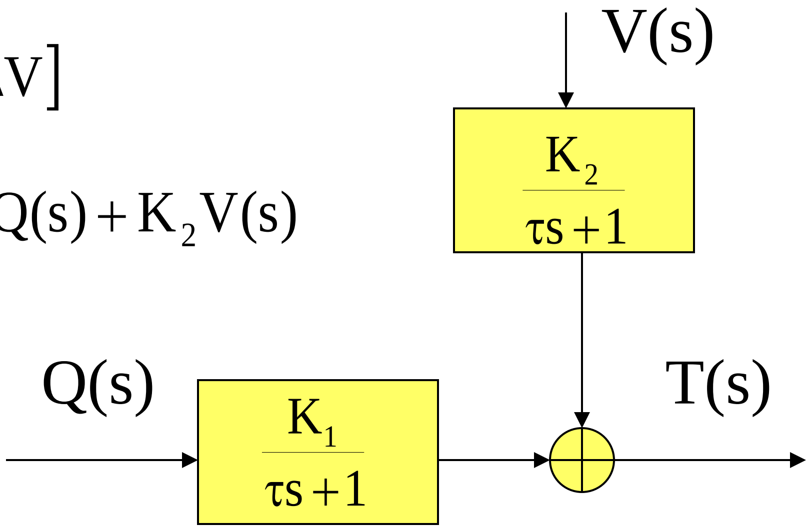
$$\tau \frac{d\Delta T}{dt} + \Delta T = K_1 \Delta q + K_2 \Delta V$$

Taking Laplace transforms with zero IC:

$$L\left[\tau \frac{d\Delta T}{dt} + \Delta T\right] = L\left[K_1 \Delta q + K_2 \Delta V\right]$$

$$\tau s T(s) + T(s) = T(s)(\tau s + 1) = K_1 Q(s) + K_2 V(s)$$

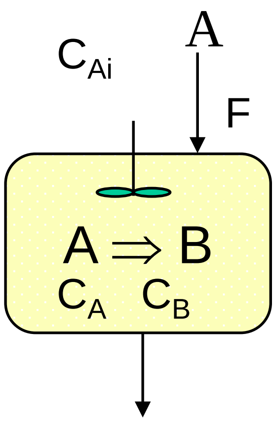
$$T(s) = \frac{K_1}{\tau s + 1} Q(s) + \frac{K_2}{\tau s + 1} V(s)$$



Isothermal Reactor. TF model

$$\frac{d\Delta c_A}{dt} = a_{11}\Delta c_A + b_{11}\Delta F + b_{12}\Delta c_{Ai}$$

$$\frac{d\Delta c_B}{dt} = a_{21}\Delta c_A + a_{22}\Delta c_B + b_{21}\Delta F$$



Taking Laplace transforms with zero IC:

$$sC_A(s) = a_{11}C_A(s) + b_{11}F(s) + b_{12}C_{Ai}(s)$$

$$C_A(s)[s - a_{11}] = b_{11}F(s) + b_{12}C_{Ai}(s)$$

$$C_A(s) = \frac{b_{11}}{s - a_{11}}F(s) + \frac{b_{12}}{s - a_{11}}C_{Ai}(s)$$

$$sC_B(s) = a_{21}C_A(s) + a_{22}C_B(s) + b_{21}F(s)$$

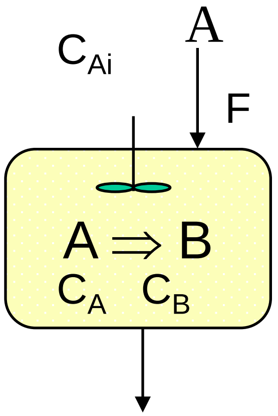
$$C_B(s)[s - a_{22}] = a_{21}C_A(s) + b_{21}F(s)$$

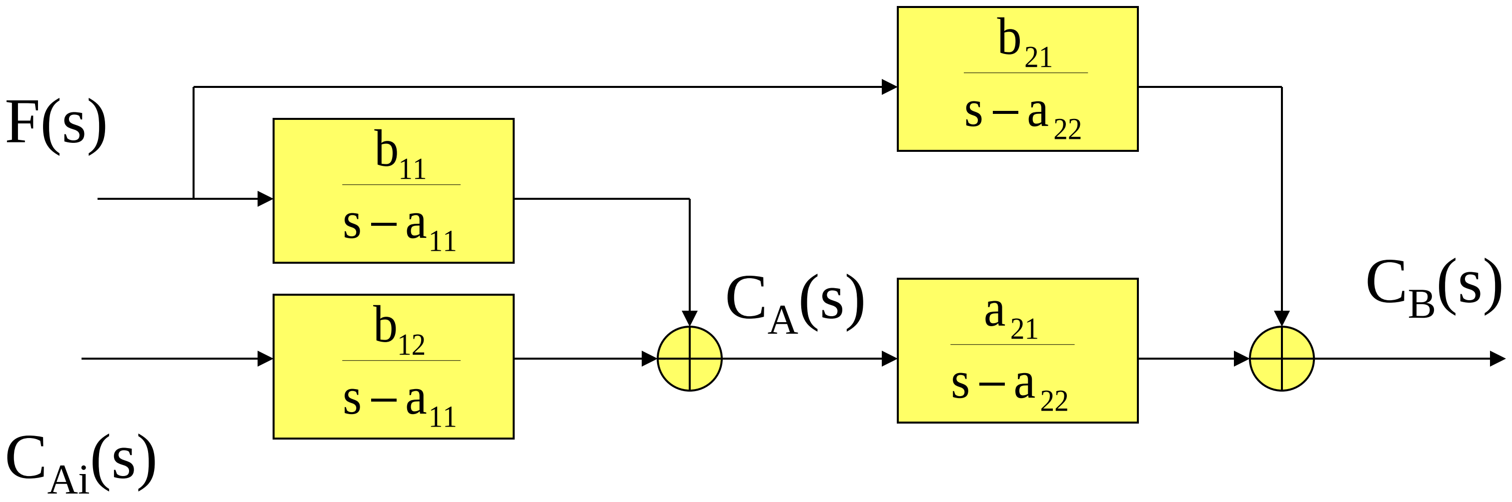
$$C_B(s) = \frac{a_{21}}{s - a_{22}}C_A(s) + \frac{b_{21}}{s - a_{22}}F(s)$$

Block Diagram

$$C_A(s) = \frac{b_{11}}{s - a_{11}} F(s) + \frac{b_{12}}{s - a_{11}} C_{Ai}(s)$$

$$C_B(s) = \frac{a_{21}}{s - a_{22}} C_A(s) + \frac{b_{21}}{s - a_{22}} F(s)$$



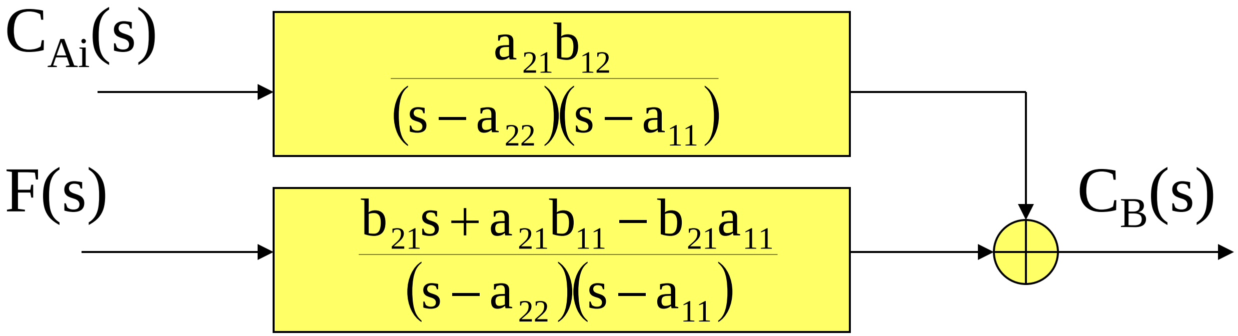


Block Diagram

$$C_B(s) = \frac{a_{21}}{s-a_{22}}\left[\frac{b_{11}}{s-a_{11}}F(s) + \frac{b_{12}}{s-a_{11}}C_{Ai}(s)\right] + \frac{b_{21}}{s-a_{22}}F(s) =$$

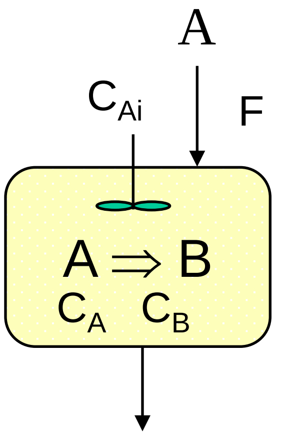
$$= \left[\frac{a_{21}}{s-a_{22}}\frac{b_{11}}{s-a_{11}} + \frac{b_{21}}{s-a_{22}}\right]F(s) + \frac{a_{21}}{s-a_{22}}\frac{b_{12}}{s-a_{11}}C_{Ai}(s) =$$

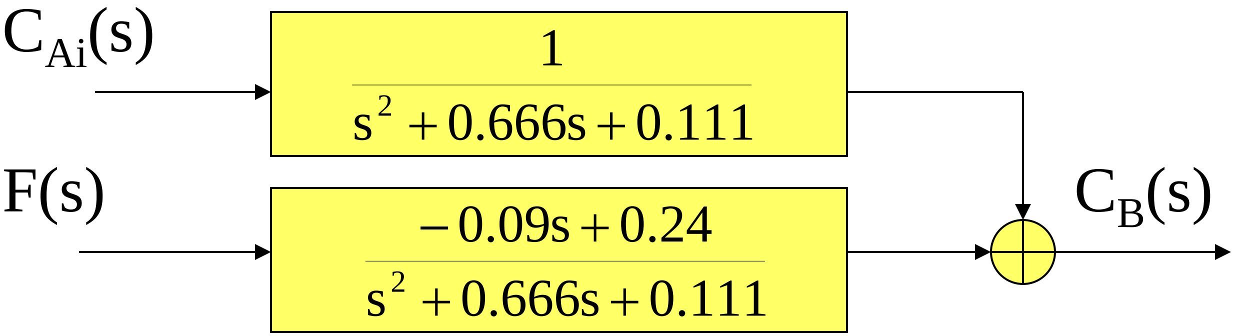
$$= \frac{b_{21}s + a_{21}b_{11} - b_{21}a_{11}}{(s-a_{22})(s-a_{11})}F(s) + \frac{a_{21}b_{12}}{(s-a_{22})(s-a_{11})}C_{Ai}(s)$$



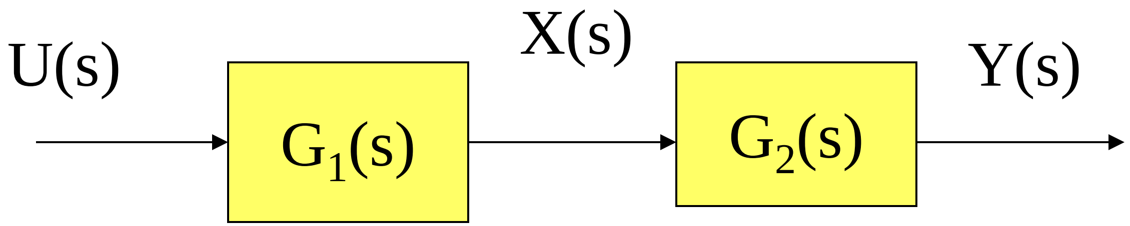
Isothermal Reactor

$$\begin{bmatrix} \frac{d\Delta c_A}{dt} \\ \frac{d\Delta c_B}{dt} \end{bmatrix} = \begin{pmatrix} -0.33 & 0 \\ 3 & -0.33 \end{pmatrix} \begin{bmatrix} \Delta c_A \\ \Delta c_B \end{bmatrix} + \begin{pmatrix} 0.09 & 0.333 \\ -0.09 & 0 \end{pmatrix} \begin{bmatrix} \Delta F \\ \Delta c_{Ai} \end{bmatrix}$$

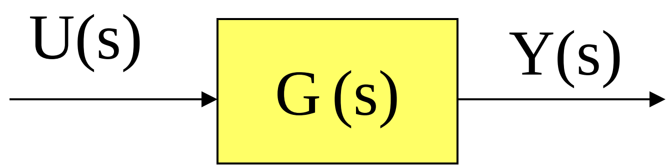




Blocks in series



$$Y(s) = G_2(s)X(s) = G_2(s)G_1(s)U(s) = G(s)U(s)$$



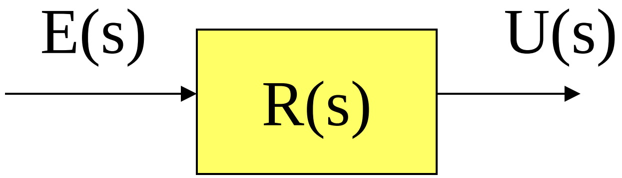
$$G(s) = G_2(s)G_1(s)$$

Transfer function of a PID

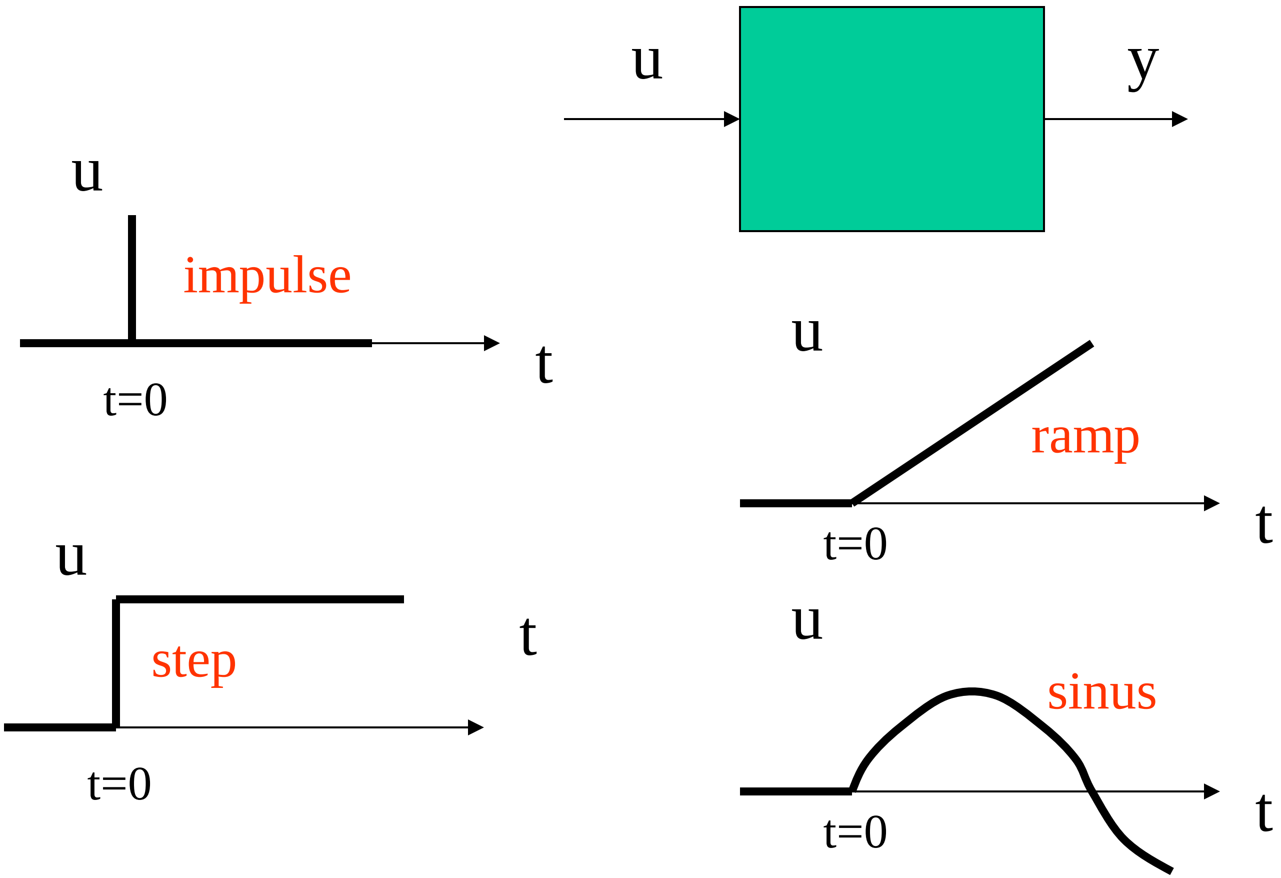
$$u(t) = K_p\left(e(t) + \frac{1}{T_i}\int_0^t e(\tau)d\tau + T_d\frac{de(t)}{dt}\right)$$

$$U(s) = K_p\left(E(s) + \frac{1}{T_i s}E(s) + T_d sE(s)\right) = K_p\left(1 + \frac{1}{T_i s} + T_d s\right)E(s)$$

$$U(s) = K_p\frac{T_d T_i s^2 + T_i s + 1}{T_i s}E(s) = R(s)E(s)$$



Normalized inputs



Poles and zeros

$$G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0} = \frac{N(s)}{D(s)}$$

Zeros of G(s) = roots of N(s) = 0

Poles of G(s) = roots of D(s) = 0

$$G(s) = \frac{s-3}{s^2 + 3s + 1} = \frac{s-3}{(s+2.618)(s+0.382)}$$

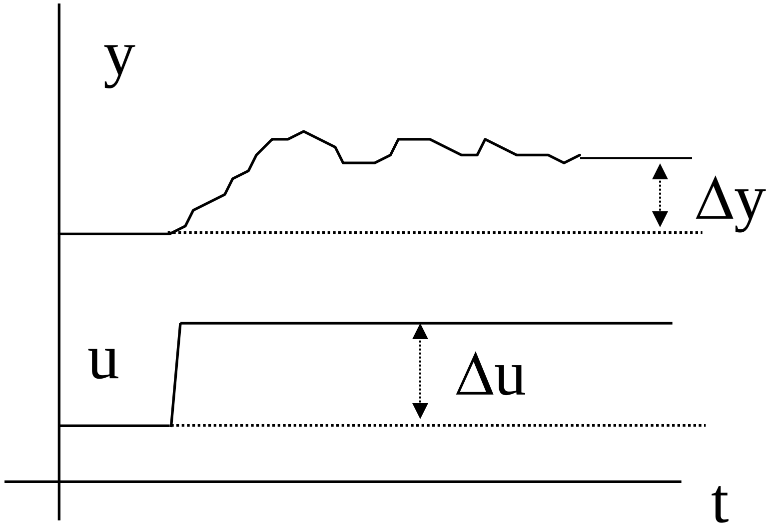
$$s - 3 = 0 \quad \text{zero in } s = 3$$

$$s^2 + 3s + 1 = 0 \quad \text{poles in } s = -2.618, \ -0.382$$

Why poles (and zeros) are important?

- As we will see later on, the type of time response of a systems when its input is changed will depend on the locations of the poles (and zeros) of the TF model.
- In the same way, the stability of the system is linked to the locations of its poles.

Gain



$$K = \left.\frac{\Delta y}{\Delta u}\right|_{\text{in equilibrium}}$$

$$K = \lim_{s \to 0} \frac{sY(s)}{sU(s)} = G(0)$$

$$G(s) = \frac{K(\beta_1 s + 1)......(\beta_m s + 1)}{(\tau_1 s + 1)(\tau_2 s + 1)......(\tau_n s + 1)}$$

format poles $-\frac{1}{\tau}$, zeros $-\frac{1}{\beta}$ and gain K. τ time constant

Poles and eigenvalues

$$G(s) = C[sI - A]^{-1}B = \frac{N(s)}{D(s)}$$

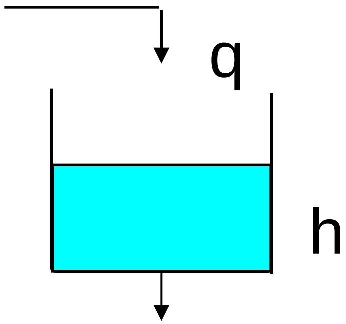
$$G(s) = C[sI - A]^{-1}B = C\frac{\text{adj}[sI - A]}{\det[sI - A]}B$$

Poles: roots of D(s) = 0

Eigenvalues: roots of $\det[sI - A] = 0$

Eigenvalues of A = poles of G(s)
(except pole/zero cancelations)

Physical Existence



Continuous physical system

It always exists

$$G(s) = \frac{K}{\tau s + 1}$$

Given a transfer function G(s)

It is possible to build a physical system which transfer function be G(s)?

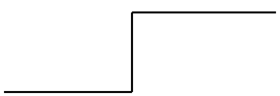
Realizability

$$G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0} = \frac{N(s)}{D(s)}$$

In order G(s) to be physically implementable: $m \le n$

If not:

$$Y(s) = \frac{s^2 + 2s + 1}{s + 2} U(s) = \left(s + \frac{1}{s + 2} \right) U(s)$$

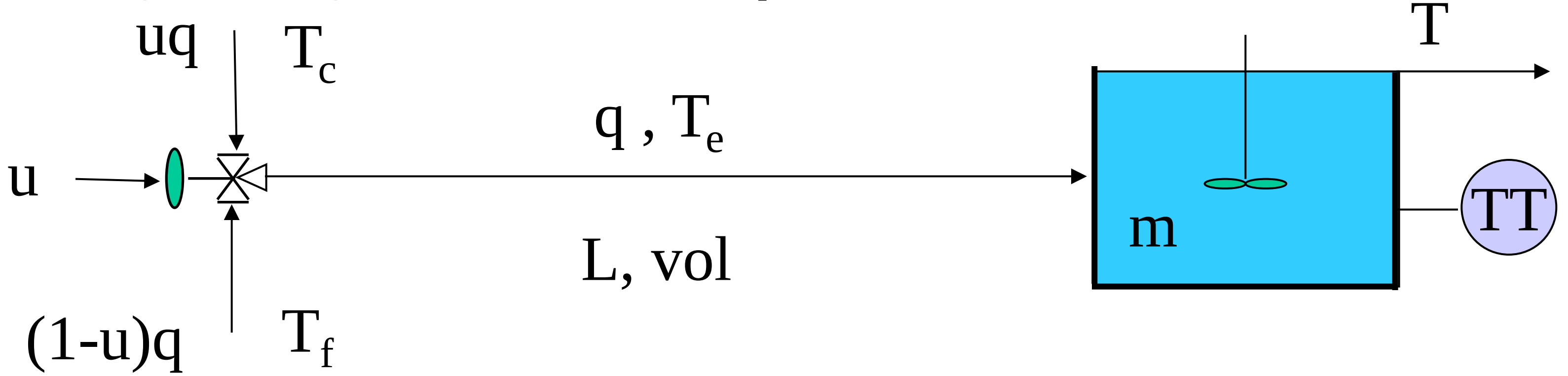


$$y(t) = \frac{du(t)}{dt} + L^{-1}\left[\frac{1}{s + 2} U(s) \right]$$

For a step change in u(t) the system should respond with an infinite y(t): This is not possible

A process with (transport) delay

u: opening in the range 0 - 1 Total flow q is constant



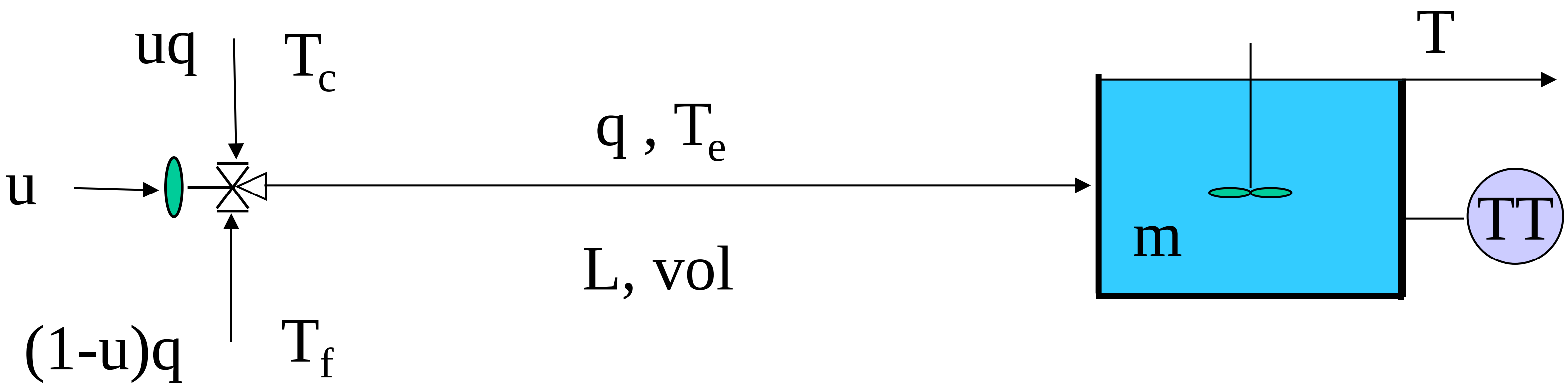
$$q\rho c_e T_e(t) = u(t) q\rho c_e T_c + (1-u(t)) q\rho c_e T_f \Rightarrow T_e(t) = u(t)T_c + (1-u(t))T_f$$

$$\frac{dV\rho c_e T(t)}{dt} = q\rho c_e T_e(t-\tau) - q\rho c_e T(t) \qquad \tau = \frac{L}{v} = \frac{LA}{vA} = \frac{vol}{q}$$

$$\frac{V}{q}\frac{dT(t)}{dt} = (T_c - T_f)u(t-\tau) + T_f - T(t)$$

Assuming constant ρ, c_e

Mixture with delay



$$\frac{V}{q}\frac{dT(t)}{dt} = (T_c - T_f)u(t-\tau) + T_f - T(t)$$

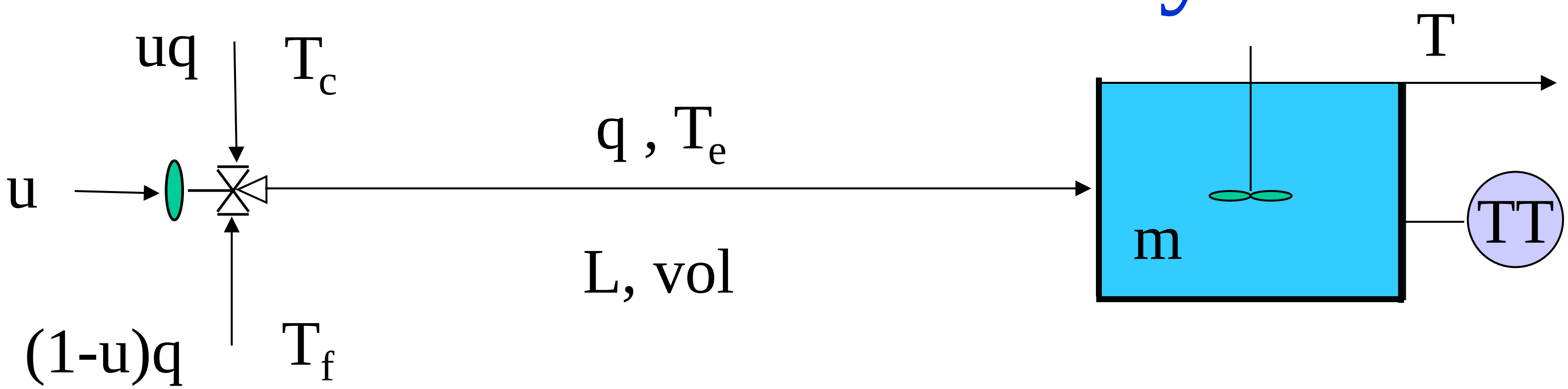
$$\frac{V}{q}\frac{dT_0}{dt} = (T_c - T_f)u_0 + T_f - T_0$$

T_0 , u_0 steady operating point

$$\frac{V}{q}\frac{d\Delta T(t)}{dt} = (T_c - T_f)\Delta u(t-\tau) - \Delta T(t)$$

$$\Delta T(t) = T(t) - T_0 \qquad \Delta u(t) = u(t) - u_0$$

Mixture with delay



$$\frac{V}{q}\frac{d\Delta T(t)}{dt} + \Delta T(t) = (T_c - T_f)\Delta u(t-\tau) \Rightarrow T(s) = \frac{e^{-\tau s}(T_c - T_f)}{\frac{V}{q}s + 1}U(s)$$

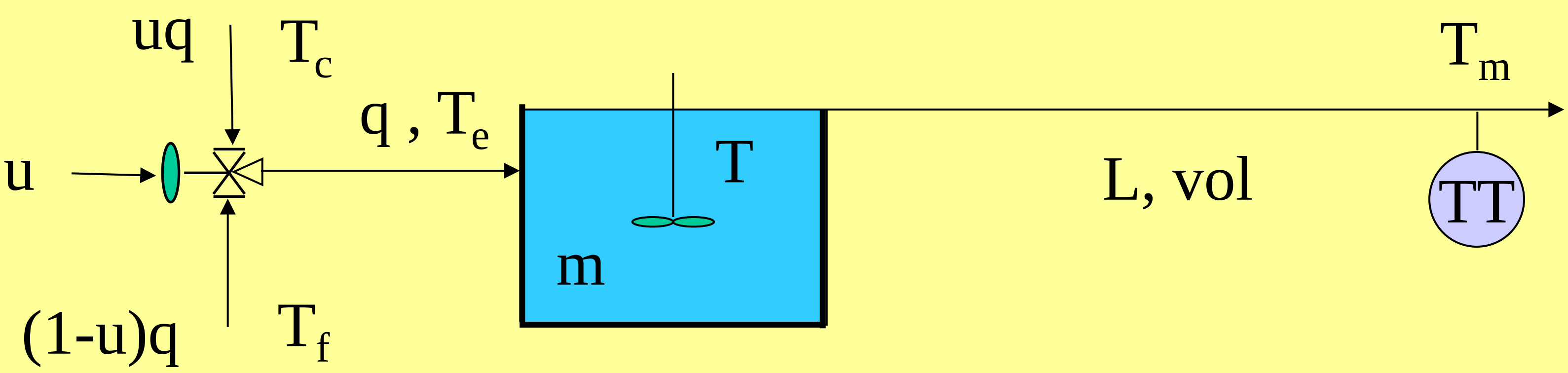
$$\frac{d\Delta T(t)}{dt} = -\frac{q}{V}\Delta T(t) + \frac{q(T_c - T_f)}{V}\Delta u(t-\tau) \qquad \Delta T(t) = 1.\Delta T(t)$$

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t-\tau)$$

$$y(t) = Cx(t)$$

Model with input delay

Output delay



$$\frac{V}{q}\frac{d\,\Delta T(t)}{dt} + \Delta T(t) = (T_c - T_f)\Delta u(t) \qquad \Delta T_m(t+\tau) = \Delta T(t)$$

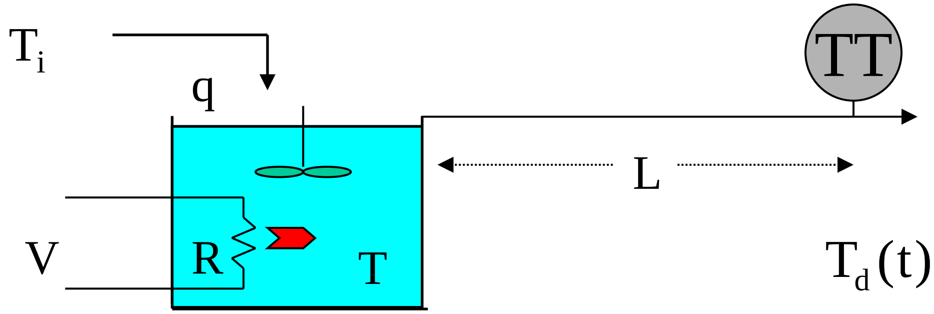
$$\frac{d\,\Delta T(t)}{dt} = -\frac{q}{V}\Delta T(t) + \frac{q(T_c - T_f)}{V}\Delta u(t) \qquad \Delta T_m(t+\tau) = 1.\Delta T(t)$$

$$\frac{d\,x(t)}{dt} = Ax(t) + Bu(t)$$

$$y(t+\tau) = Cx(t)$$

Model with output delay

Delay

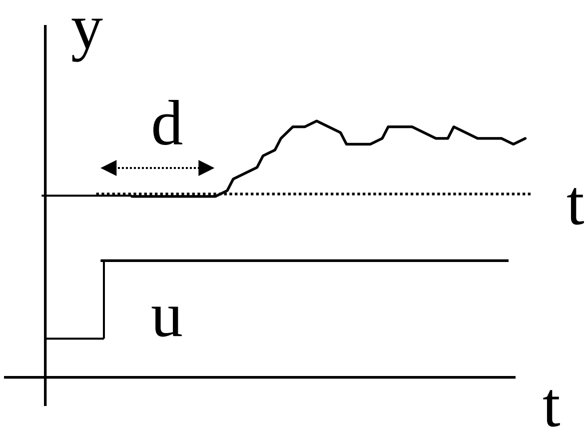


$$T_d(t) = T(t-d) = T(t - \frac{L}{v})$$

$$T(s) = \frac{K_1}{\tau_1 s + 1} Q(s) + \frac{K_2}{\tau_1 s + 1} V(s)$$

$$T_d(s) = e^{-ds} T(s) = \frac{e^{-ds} K_1}{\tau_1 s + 1} Q(s) + \frac{e^{-ds} K_2}{\tau_1 s + 1} V(s)$$

$$G(s) = \frac{e^{-ds} K(\beta_1 s + 1)......(\beta_m s + 1)}{(\tau_1 s + 1)(\tau_2 s + 1)......(\tau_n s + 1)}$$



Pade Approximation

$$G(s) = \frac{e^{-ds} K(\beta_1 s + 1)......(\beta_m s + 1)}{(\tau_1 s + 1)(\tau_2 s + 1)......(\tau_n s + 1)}$$

G(s) with a delay d is not a rational function. If required, the delay can be approximated by a Taylor series expansion:

2º order approx.:

$$e^{-ds} \approx \frac{1 - \frac{ds}{2} + \frac{(ds)^2}{12}}{1 + \frac{d}{2}s + \frac{(ds)^2}{12}}$$

$$e^{-ds} \approx \frac{1 - \frac{d}{2}s}{1 + \frac{d}{2}s}$$

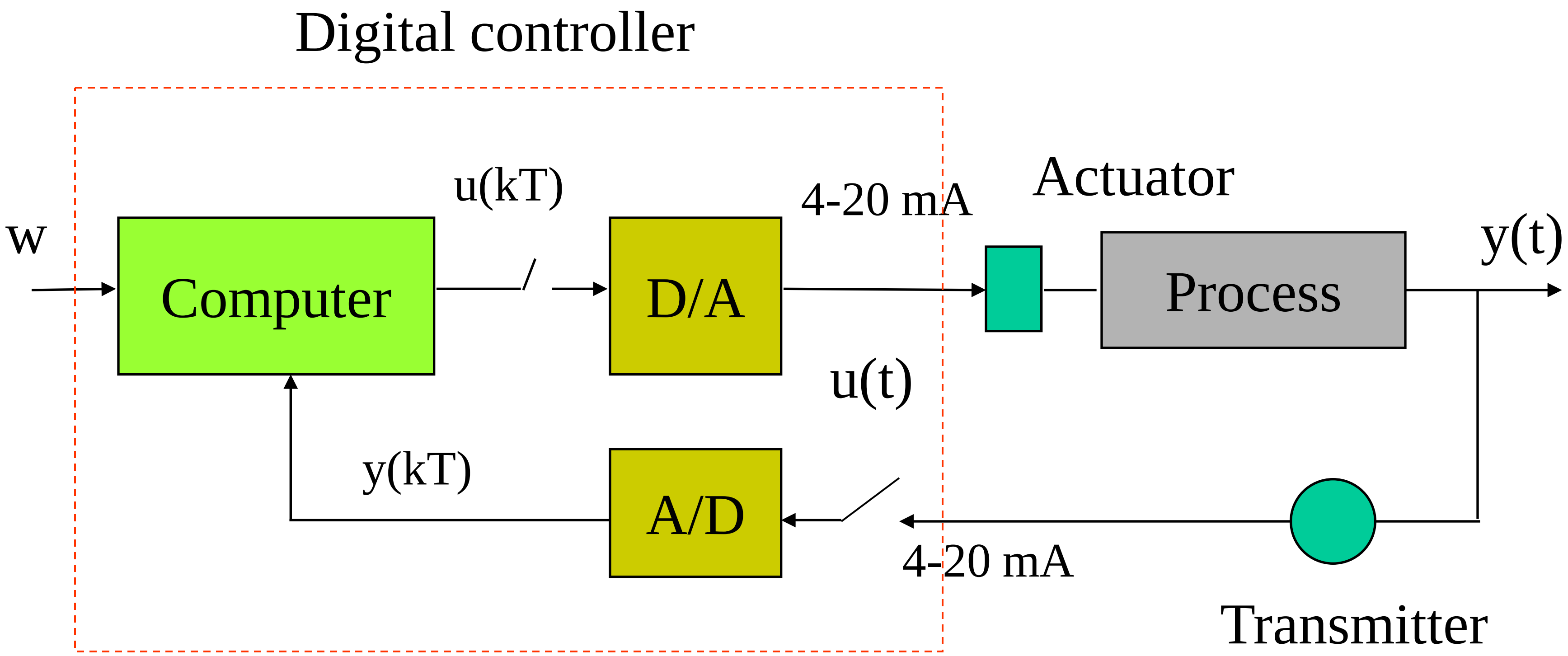
First order Pade approximation

resppade

Why studying mathematical models?

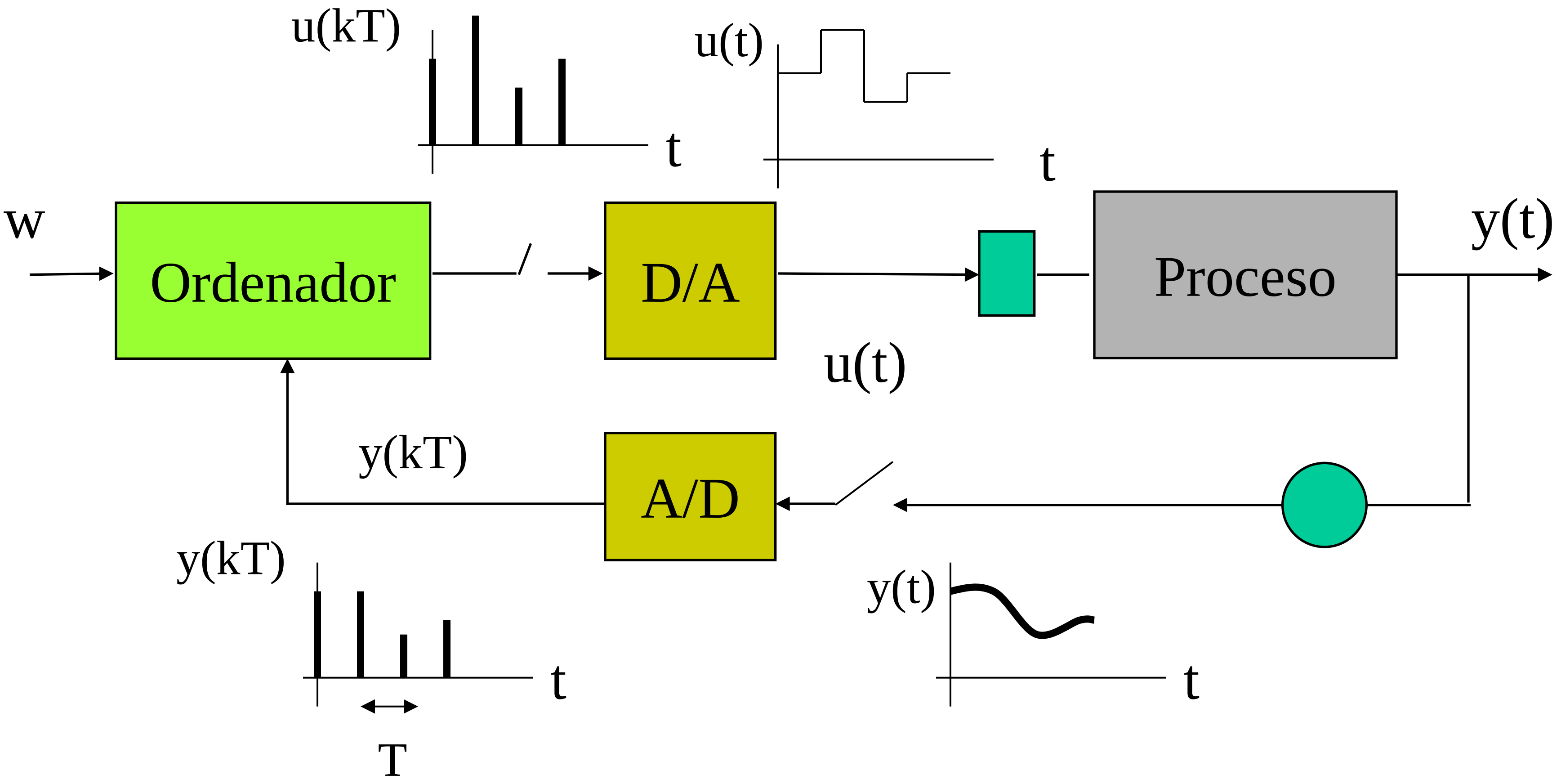
- We can predict the response of the system to an action from the mathematical model
- Many dynamic properties of the processes depend on their mathematical model estructure. Analysing dynamics requires to use a model
- Closed loop dynamics can be completely different to the open loop one. We need mathematical models to predict it.
- Desingning good performance and robust control system requires a model

Computer controlled processes



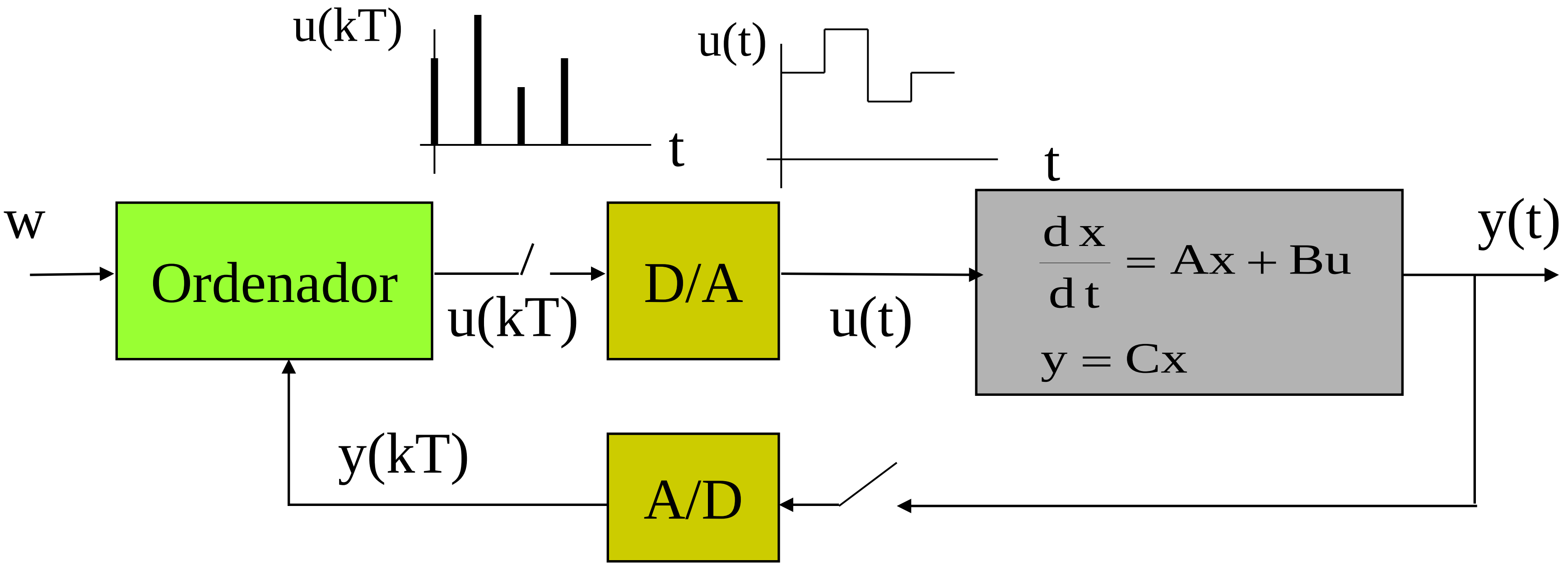
The signals received and processed by the computer are digital, not analog ones, and only change at the sampling times.

Signals



The information in the computer is updated every T time units (the sampling period)

Sampled-data model



Find a model y(kT) = f(u(kT)) such that y(kT) = y(t) at the sampling times

Sampled-data model

$$\frac{dx}{dt} = Ax + Bu$$

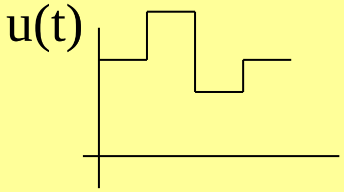
$$y = Cx + Du$$

$$x(t) = e^{A(t-t_0)} x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)} Bu(\tau) d\tau$$

Taking as initial and final times the instants kT and (k+1)T of a sampling period:

$$x((k+1)T) = e^{AT} x(kT) + \int_{kT}^{(k+1)T} e^{A((k+1)T-\sigma)} Bu(\sigma) d\sigma$$

Sampled-data model



For a sampling period, u(t) is constant and equal to u(kT)

$$x((k+1)T) = e^{AT}x(kT) + \int_{kT}^{(k+1)T} e^{A((k+1)T-\sigma)}Bu(\sigma)d\sigma =$$

$$= e^{AT}x(kT) + \int_{kT}^{(k+1)T} e^{A((k+1)T-\sigma)}d\sigma \, Bu(kT)$$

$$\text{change of variable}: \quad \tau = (k+1)T - \sigma, \quad d\tau = -d\sigma$$

$$x((k+1)T) = e^{AT}x(kT) + \int_{0}^{T} e^{A\tau}d\tau \, Bu(kT)$$

Sampled-data model

$$\frac{dx}{dt} = Ax + Bu$$

$$y = Cx + Du$$

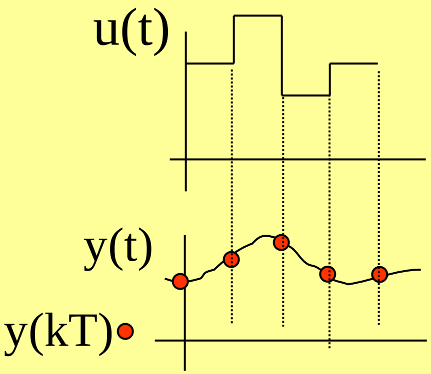
$$x((k+1)T) = \Phi x(kT) + \Gamma u(kT)$$

$$y(kT) = Cx(kT)$$

$$\Phi = e^{AT} \qquad \Gamma = \int_0^T e^{A\tau} d\tau \, B$$

Matlab c2d

Equation in differences



With this type of inputs, the discretized model provides the same values at the time instants t = kT than the continuous model (Starting from the same initial state and applying the same inputs)

Sampled-data model

$$\frac{dx}{dt} = Ax + Bu$$

$$y = Cx$$

$$x((k+1)T) = \Phi x(kT) + \Gamma u(kT)$$

$$y(kT) = Cx(kT)$$

$$\Phi = e^{AT} \qquad \Gamma = \int_0^T e^{A\tau} d\tau \, B$$

Simplified notation:

k refers to the first, second third, etc. sampling time

$$x(k+1) = \Phi x(k-1) + \Gamma u(k-1)$$

$$y(k) = Cx(k)$$

Example: Tank

If $\Delta q = 0$:

$$\frac{d\Delta h}{dt} = \alpha \Delta h + \beta \Delta u$$

$$\Delta h = 1.\Delta h$$

$$x((k+1)T) = \Phi x(kT) + \Gamma u(kT)$$

$$y(kT) = Cx(kT)$$

$$\Phi = e^{AT} \quad \Gamma = \int_0^T e^{A\tau} d\tau\, B$$

$$\Phi = e^{\alpha T} \quad \Gamma = \int_0^T e^{\alpha\tau} d\tau\, \beta = \frac{\beta}{\alpha}(e^{\alpha T} - 1)$$

$$\Delta h((k+1)T) = e^{\alpha T} \Delta h(kT) + \frac{\beta}{\alpha}(e^{\alpha T} - 1)\Delta u(kT)$$

Discretized model: equation in differences

Example: Tank

If $\Delta q = 0$:

$$\frac{d\Delta h}{dt} = \alpha\Delta h + \beta\Delta u$$

$$\Delta h = 1.\Delta h$$

Si

$$\alpha = \frac{-u_0 k}{2A\sqrt{h_0}} = -1.252$$

$$\beta = \frac{-k\sqrt{h_0}}{A} = -0.167$$

$$T = 0.5$$

$$x((k+1)T) = \Phi x(kT) + \Gamma u(kT)$$

$$y(kT) = Cx(kT)$$

$$\Delta h((k+1)T) = e^{\alpha T}\Delta h(kT) + \frac{\beta}{\alpha}(e^{\alpha T} - 1)\Delta u(kT)$$

$$\boxed{\Delta h((k+1)0.5) = 0.535\Delta h(k0.5) - 0.062\Delta u(k0.5)}$$

Discretized model: equation in differences

Time response

$$x(k+1) = \Phi x(k) + \Gamma u(k)$$
$$y(k) = Cx(k)$$

Initial conditions: x(0)

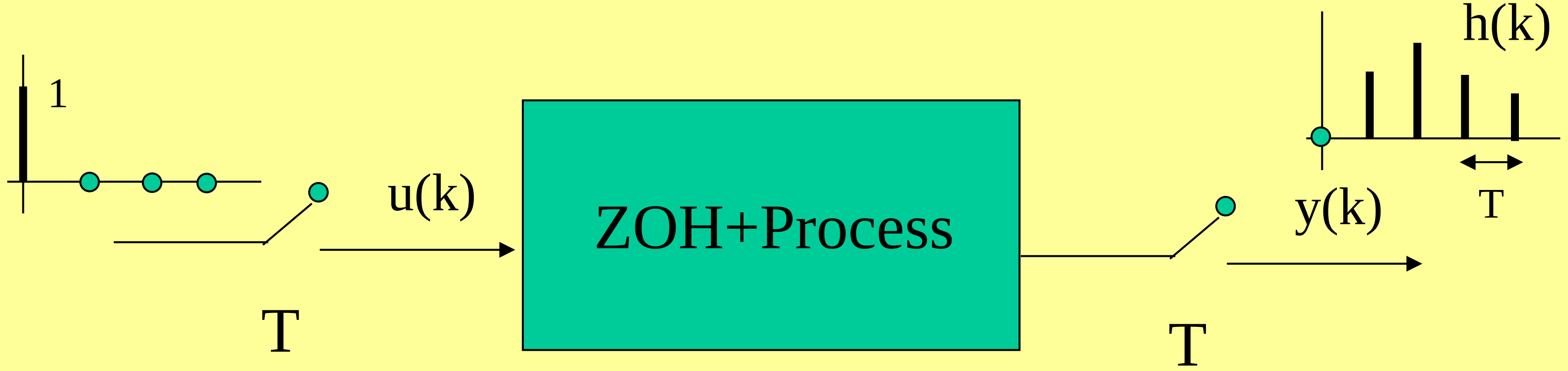
$$x(1) = \Phi x(0) + \Gamma u(0)$$
$$x(2) = \Phi x(1) + \Gamma u(1) = \Phi\left[\Phi x(0) + \Gamma u(0)\right] + \Gamma u(1) =$$
$$= \Phi^2 x(0) + \Phi\Gamma u(0) + \Gamma u(1)$$
$$x(3) = \Phi x(2) + \Gamma u(2) = \Phi\left[\Phi^2 x(0) + \Phi\Gamma u(0) + \Gamma u(1)\right] + \Gamma u(2) =$$
$$= \Phi^3 x(0) + \Phi^2\Gamma u(0) + \Phi\Gamma u(1) + \Gamma u(2)$$

....

$$x(k) = \Phi^k x(0) + \sum_{i=0}^{k-1} \Phi^{k-i-1}\Gamma u(i) \qquad y(k) = C\Phi^k x(0) + \sum_{i=0}^{k-1} C\Phi^{k-i-1}\Gamma u(i)$$

Pulsed impulse response

$$y(k) = C\Phi^k x(0) + \sum_{i=0}^{k-1} C\Phi^{k-i-1}\Gamma u(i)$$



Unity input at t = 0

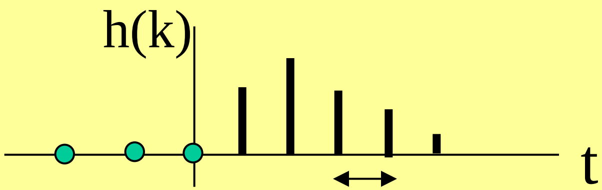
Response starting from zero ic.

$$y(k) = C\Phi^k x(0) + \sum_{i=0}^{k-1} C\Phi^{k-i-1}\Gamma u(i) = C\Phi^{k-1}\Gamma = h(k)$$

$$y(k) = \sum_{i=0}^{k-1} h(k-i)u(i)$$

Impulse response model

Impulse response model



$$y(k) = \sum_{i=0}^{k-1} h(k-i)u(i) =$$

$$= h(k)u(0) + h(k-1)u(1) + \ldots + h(2)u(k-2) + h(1)u(k-1) =$$

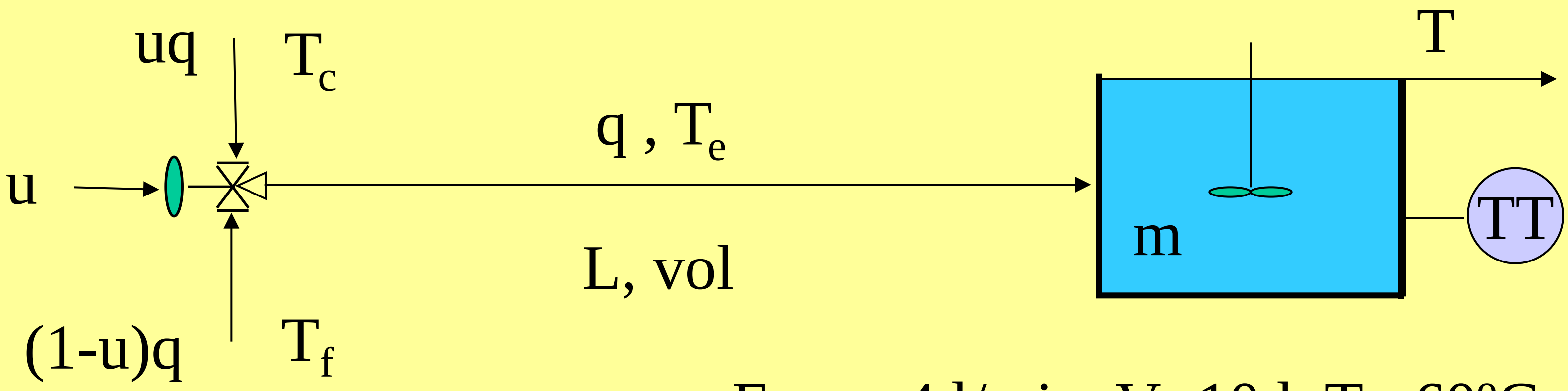
$$= \sum_{j=1}^{k} h(j)u(k-j)$$

As $h(i) = 0$ for $i \leq 0$ and with zero initial conditions: $u(i) = 0$ for $i < 0$:

$$y(k) = \sum_{i=0}^{\infty} h(k-i)u(i) = \sum_{j=1}^{\infty} h(j)u(k-j)$$

The system output is a linear combination of past input values

Example: Mixture



For q=4 l/min, V=10 l, T_c=60ºC, T_f=10ºC, vol=4 l, period = 0.5 min.

$$\frac{d\,\Delta T(t)}{dt} = -\frac{q}{V}\Delta T(t) + \frac{q(T_c - T_f)}{V}\Delta u(t-\tau) \qquad \tau = \frac{4}{4} = 1\,\text{min}$$

$$\Phi = e^{AT} = e^{-\frac{4}{20}0.5} = 0.905 \qquad \Gamma = \int_0^{0.5} e^{-\frac{4}{20}\tau} d\tau \frac{4}{20}(60-10) = 4.75$$

$$T(k+1) = 0.905T(k) + 4.75u(k-2)$$

Shift operator q^{-1}

$$q^{-1}z(k) = z(k-1) \qquad qz(k) = z(k+1)$$

$$x(k+1) = qx(k) = \Phi x(k) + \Gamma u(k)$$

$$[qI - \Phi]x(k) = \Gamma u(k)$$

$$x(k) = [qI - \Phi]^{-1}\Gamma u(k)$$

$$y(k) = C[qI - \Phi]^{-1}\Gamma u(k)$$

$$\frac{y(k)}{u(k)} = C[qI - \Phi]^{-1}\Gamma = \frac{b_0 q^m + b_1 q^{m-1} + \ldots + b_{m-1}q^1 + b_m}{q^n + a_1 q^{n-1} + \ldots + a_{n-1}q^1 + a_n}$$

Rational function of q

Pulsed transfer function

$$y(k) = C\left[qI - \Phi\right]^{-1}\Gamma u(k) = \frac{b_0 q^m + b_1 q^{m-1} + \ldots + b_{m-1} q^1 + b_m}{q^n + a_1 q^{n-1} + \ldots + a_{n-1} q^1 + a_n} u(k) =$$

$$= \frac{q^{-n}[b_0 q^m + b_1 q^{m-1} + \ldots + b_{m-1} q^1 + b_m]}{q^{-n}[q^n + a_1 q^{n-1} + \ldots + a_{n-1} q^1 + a_n]} u(k) =$$

$$= \frac{q^{-(n-m)}(b_0 + b_1 q^{-1} + \ldots + b_{m-1} q^{-m+1} + b_m q^{-m})}{1 + a_1 q^{-1} + \ldots + a_{n-1} q^{-n+1} + a_n q^{-n}} u(k)$$

$$d = n - m$$

$$y(k) = \frac{B(q^{-1})}{A(q^{-1})} u(k) = \frac{q^{-d}(b_0 + b_1 q^{-1} + \ldots + b_{m-1} q^{-m+1} + b_m q^{-m})}{1 + a_1 q^{-1} + \ldots + a_{n-1} q^{-n+1} + a_n q^{-n}} u(k)$$

Pulsed transfer function

$$y(k) = \frac{B(q^{-1})}{A(q^{-1})}u(k) = \frac{q^{-d}(b_0 + b_1 q^{-1} + \ldots + b_m q^{-m})}{1 + a_1 q^{-1} + a_2 q^{-2} + \ldots + a_n q^{-n}}u(k)$$

$$A(q^{-1})y(k) = B(q^{-1})u(k)$$

$$(1 + a_1 q^{-1} + a_2 q^{-2} \ldots + a_n q^{-n})y(k) = q^{-d}(b_0 + b_1 q^{-1} + \ldots + b_m q^{-m})u(k)$$

$$y(k) + a_1 y(k-1) + a_2 y(k-2) + \ldots + a_n y(k-n) =$$

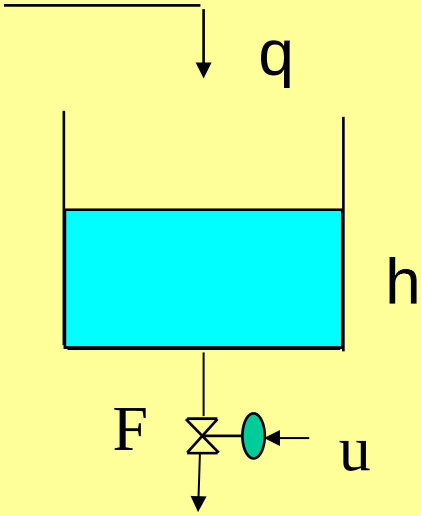
$$b_0 u(k-d) + b_1 u(k-d-1) + \ldots + b_m u(k-d-m)$$

$$y(k) = -a_1 y(k-1) - a_2 y(k-2) - \ldots - a_n y(k-n) +$$

$$+ b_0 u(k-d) + b_1 u(k-d-1) + \ldots + b_m u(k-d-m)$$

The system output is a linear combination of past output and input values

Example: tank



$$\Delta h((k+1)0.5) = 0.535\Delta h(k0.5) - 0.062\Delta u(k0.5)$$

$$y(k) = \frac{B(q^{-1})}{A(q^{-1})}u(k) = C\left[qI - \Phi\right]^{-1}\Gamma u(k) =$$

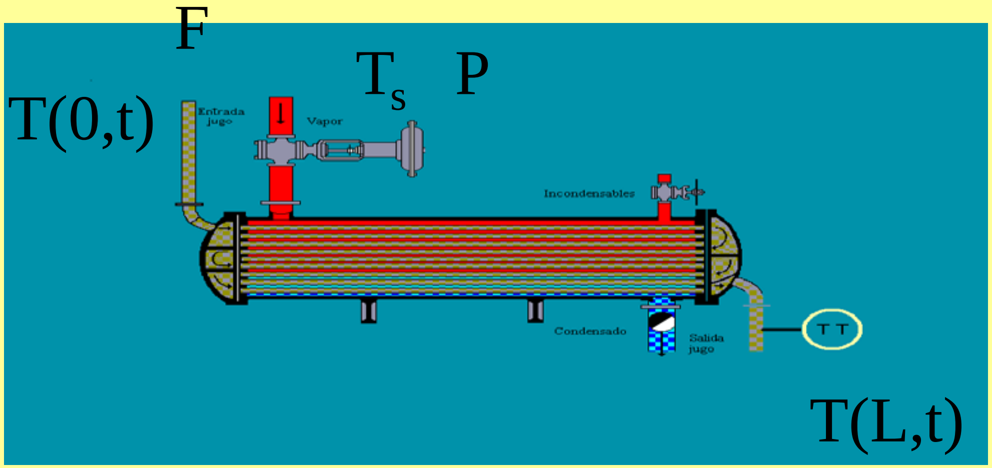
$$= 1\left[q - 0.535\right]^{-1}(-0.062)u(k) =$$

T = 0.5

$$= \frac{-0.062}{q - 0.535}u(k) = \frac{-0.062q^{-1}}{1 - 0.535q^{-1}}u(k)$$

Pole = Eigenvalue = 0.535

Distributed parameters process



Partial differential equation

$$\frac{\partial T(x,t)}{\partial t} = -\frac{F}{\pi r^2}\frac{\partial T(x,t)}{\partial x} + \frac{2U(T_s(t) - T(x,t))}{r\rho c_e}$$

Distributed parameters process

For F = cte.

$$\frac{\partial T(x,t)}{\partial t} = -\frac{F}{\pi r^2}\frac{\partial T(x,t)}{\partial x} + \frac{2U(T_s(t) - T(x,t))}{r\rho c_e}$$

In equilibrium:

$$0 = -\frac{F}{\pi r^2}\frac{\partial \overline{T}(x,t)}{\partial x} + \frac{2U(\overline{T}_s - \overline{T}(x,t))}{r\rho c_e}$$

In terms of the deviations ΔT over the equilibrium:

$$\frac{\partial \Delta T(x,t)}{\partial t} = -\frac{F}{\pi r^2}\frac{\partial \Delta T(x,t)}{\partial x} + \frac{2U(\Delta T_s(t) - \Delta T(x,t))}{r\rho c_e}$$

$$\frac{\partial \Delta T(x,t)}{\partial t} = -\alpha\frac{\partial \Delta T(x,t)}{\partial x} + \beta(\Delta T_s(t) - \Delta T(x,t))$$

s Laplace transform with respect to t

p Laplace transform with respect to x

Distributed parameters process

$$\frac{\partial \Delta T(x,t)}{\partial t} = -\alpha \frac{\partial \Delta T(x,t)}{\partial x} + \beta(\Delta T_s(t) - \Delta T(x,t))$$

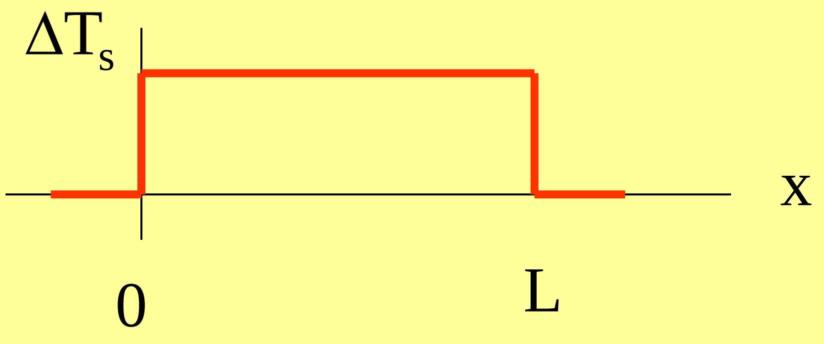
Laplace transform s with respect to t:

$$s\Delta T(x,s) = -\alpha \frac{d\Delta T(x,s)}{dx} + \beta(\Delta T_s(s) - \Delta T(x,s))$$

p with respect to x:

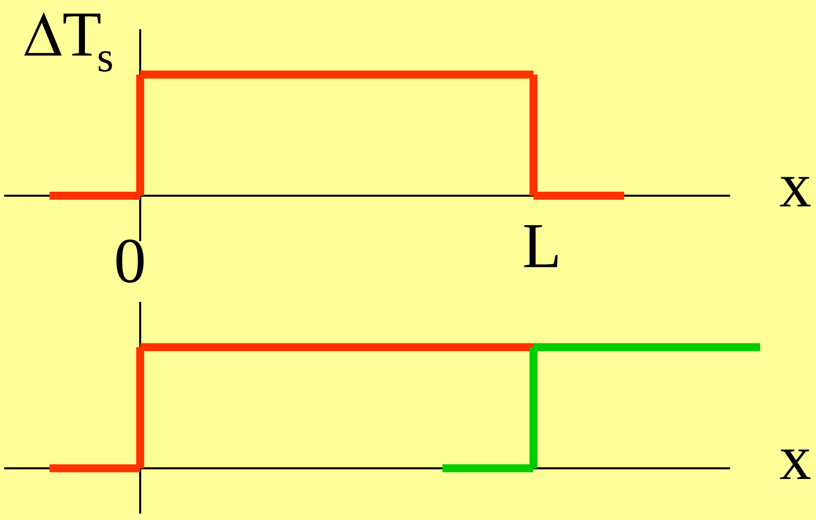
$$s\Delta T(p,s) = -\alpha p \Delta T(p,s) + \alpha \Delta T(0,s) + \beta(\Delta T_s(s,p) - \Delta T(p,s))$$

Shape of the changes of steam temperature over the heat exchanger



Distributed parameters process

This shape can be obtained as the difference between two step changes placed a distance from each other L



$$\Delta T_s(p,s) = \frac{\Delta T_s(s)}{p} - e^{-pL}\frac{\Delta T_s(s)}{p}$$

$$s\Delta T(p,s) = -\alpha p \Delta T(p,s) + \alpha \Delta T(0,s) + \beta\left(\Delta T_s(s)\frac{1-e^{-pL}}{p} - \Delta T(p,s)\right)$$

Distributed parameters process

$$s\Delta T(p,s) = -\alpha p \Delta T(p,s) + \alpha \Delta T(0,s) + \beta\left(\Delta T_s(s)\frac{1-e^{-pL}}{p} - \Delta T(p,s)\right)$$

$$(s + \alpha p + \beta)\Delta T(p,s) = \alpha \Delta T(0,s) + \beta \Delta T_s(s)\frac{1-e^{-pL}}{p}$$

$$\Delta T(p,s) = \frac{1}{p + \frac{s+\beta}{\alpha}}\Delta T(0,s) + \frac{\beta/\alpha}{p + \frac{s+\beta}{\alpha}}\frac{1-e^{-pL}}{p}\Delta T_s(s)$$

Double transfer function in p and s with respect to the changes in the input temperature and the heating steam

Distributed parameters process

$$\Delta T(p,s) = \frac{1}{p+\frac{s+\beta}{\alpha}}\Delta T(0,s) + \frac{\beta/\alpha}{p+\frac{s+\beta}{\alpha}}\frac{1-e^{-pL}}{p}\Delta T_s(s)$$

Taking the inverse Laplace transfer with respect to p: (0≤x ≤L)

$$\Delta T(x,s) = e^{-\frac{s+\beta}{\alpha}x}\frac{\alpha}{s+\beta}\Delta T(0,s) + (1-e^{-\frac{s+\beta}{\alpha}x})\frac{\beta}{s+\beta}\Delta T_s(s)$$

For x = L:

$$\Delta T(L,s) = e^{-\frac{s+\beta}{\alpha}L}\frac{\alpha}{s+\beta}\Delta T(0,s) + (1-e^{-\frac{s+\beta}{\alpha}L})\frac{\beta}{s+\beta}\Delta T_s(s)$$

Transfer Functions with respect to s: first order systems with delay

$$\Delta T(L,s) = \boxed{e^{-\frac{L}{\alpha}s}\frac{\alpha e^{-\frac{\beta L}{\alpha}}}{s+\beta}}\Delta T(0,s) + \boxed{(1-e^{\frac{-L}{\alpha}s}e^{-\frac{\beta L}{\alpha}})\frac{\beta}{s+\beta}}\Delta T_s(s)$$

T(L,s) temperature at the heat exchanger output

Distributed parameters process

V, volume of the pipes

$$\Delta T(L,s) = e^{-\frac{L}{\alpha}s} \frac{\alpha e^{-\frac{\beta L}{\alpha}}}{s+\beta} \Delta T(0,s) + (1 - e^{\frac{-L}{\alpha}s} e^{-\frac{\beta L}{\alpha}}) \frac{\beta}{s+\beta} \Delta T_s(s)$$

A surface of the pipes

$$\Delta T(L,s) = e^{-\frac{\pi r^2 L}{F}s} \frac{\alpha e^{-\frac{2U\pi r^2 L}{r\rho c_e F}}}{s+\beta} \Delta T(0,s) + (1 - e^{-\frac{\pi r^2 L}{F}s} e^{-\frac{2U\pi r^2 L}{r\rho c_e F}}) \frac{\beta}{s+\beta} \Delta T_s(s)$$

Delay V/F in the response to changes in the input temperature

$$\Delta T(L,s) = e^{-\frac{V}{F}s} \left[\frac{\rho c_e F}{U 2\pi r} \right] \frac{e^{-\frac{UA}{\rho c_e F}}}{\frac{r\rho c_e}{2U}s + 1} \Delta T(0,s) + \frac{1 - e^{-\frac{V}{F}s} e^{-\frac{UA}{\rho c_e F}}}{\frac{r\rho c_e}{2U}s + 1} \Delta T_s(s)$$