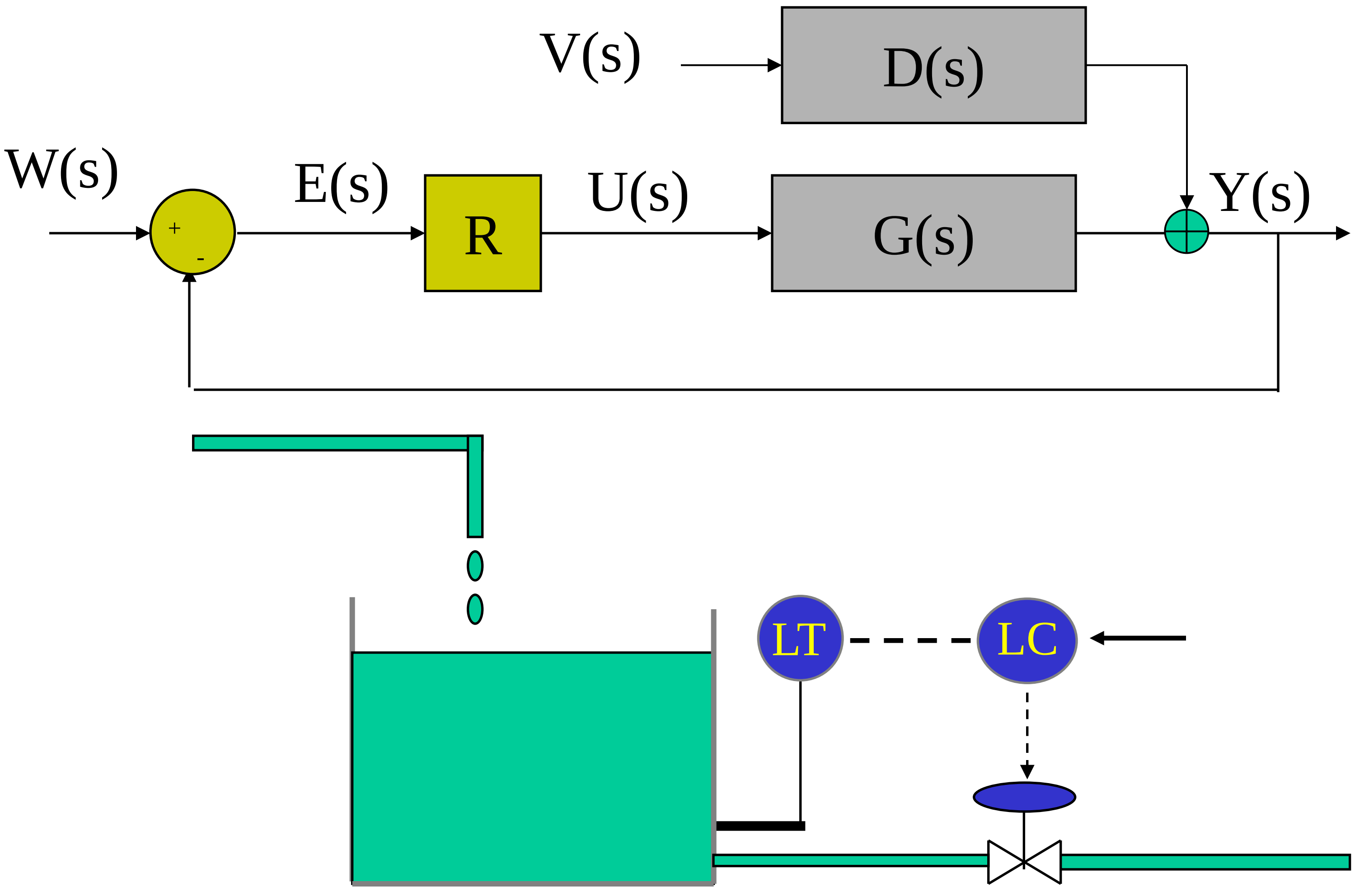
Systems Analysis

Prof. Cesar de Prada
ISA-UVA
prada@autom.uva.es

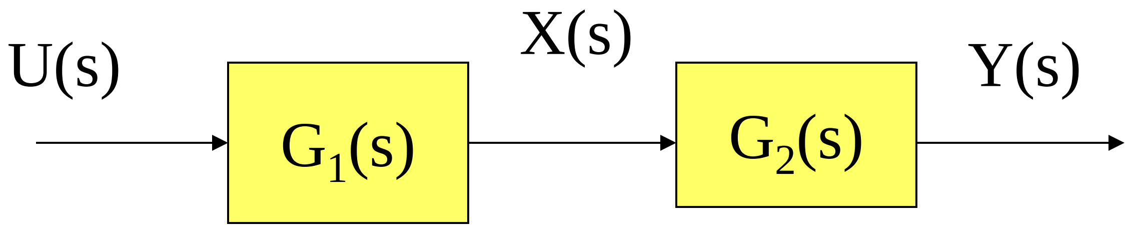
Aims

- Learn how to infer the dynamic behaviour of a closed loop system from its model.
- Learn how to infer the changes in the dynamic of a closed loop system as a function of the controller parameters.
- Be aware of the constraints imposed by process (and the controller) on the achievable performance of the closed loop system

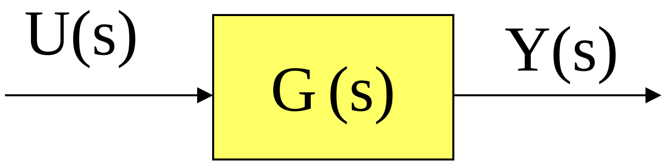
A control loop



Blocks in series

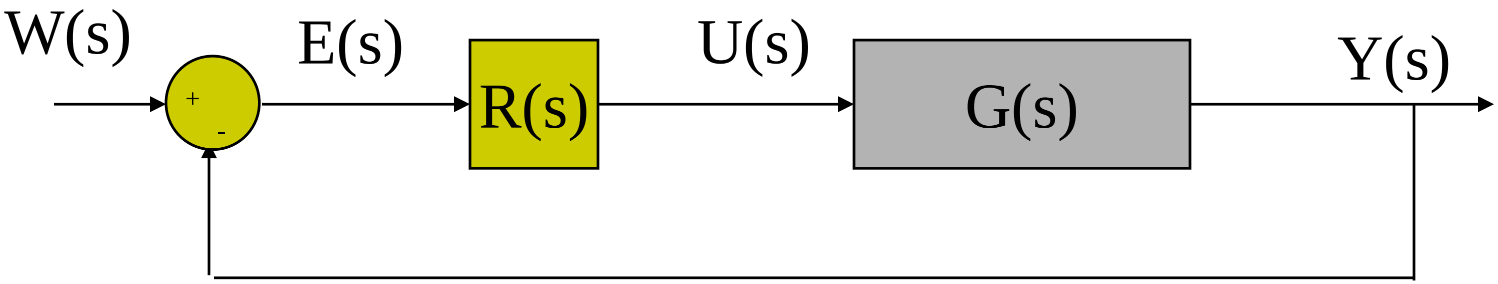


$$Y(s) = G_2(s)X(s) = G_2(s)G_1(s)U(s) = G(s)U(s)$$



$$G(s) = G_2(s)G_1(s)$$

Closed Loop Transfer Function (CLTF)

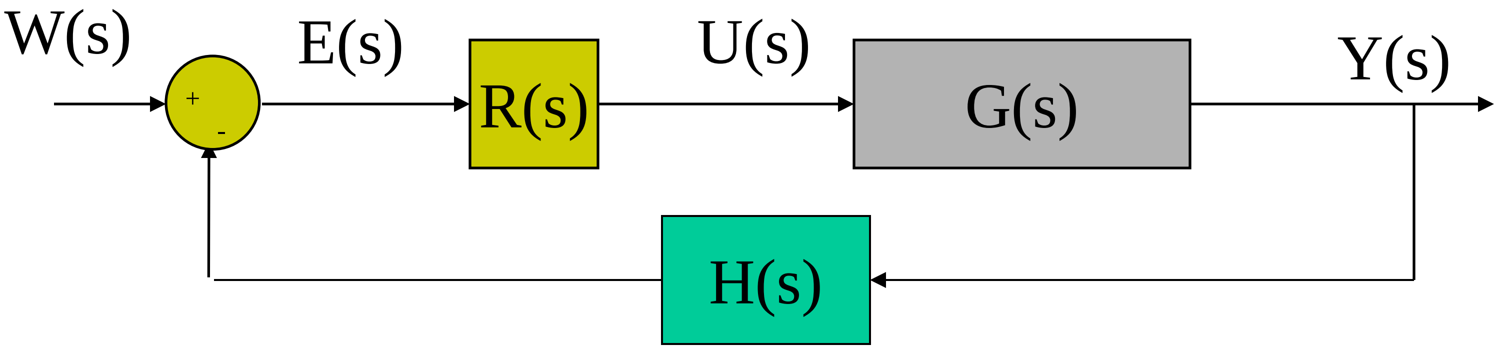


$$Y(s) = G(s)U(s) = G(s)R(s)E(s) = G(s)R(s)\left[W(s) - Y(s)\right]$$

$$Y(s)\left[1 + G(s)R(s)\right] = G(s)R(s)W(s)$$

$$Y(s) = \frac{G(s)R(s)}{1 + G(s)R(s)} W(s)$$

Closed loop systems

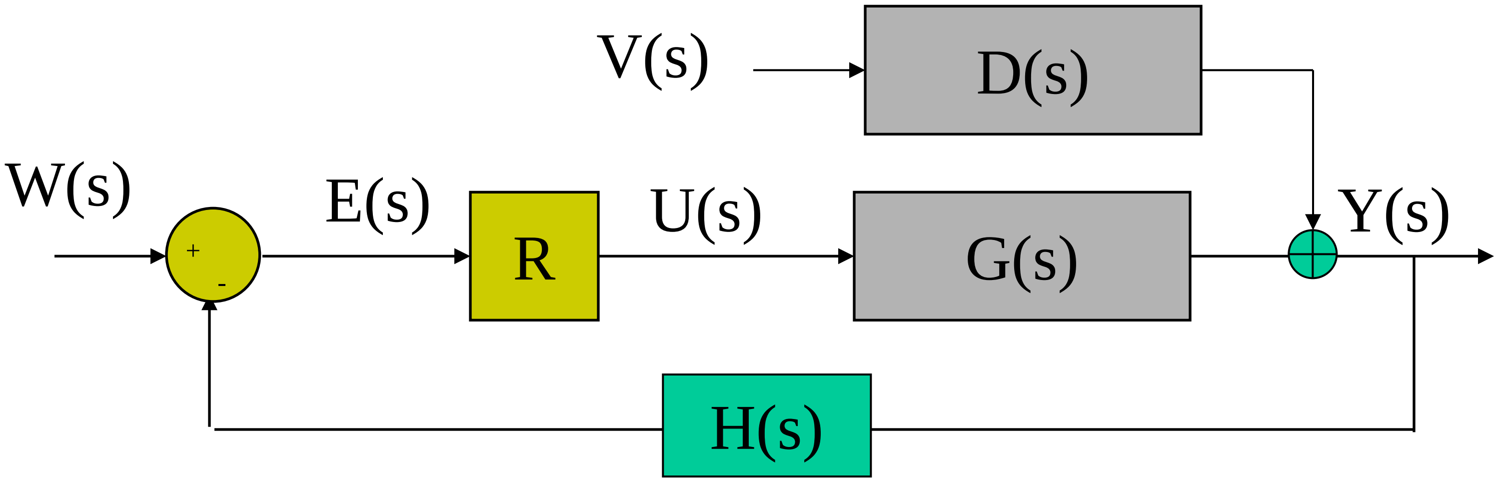


$$Y(s) = G(s)U(s) = G(s)R(s)E(s) = G(s)R(s)\left[W(s) - H(s)Y(s)\right]$$

$$Y(s)\left[1 + G(s)R(s)H(s)\right] = G(s)R(s)W(s)$$

$$Y(s) = \frac{G(s)R(s)}{1 + G(s)R(s)H(s)} W(s)$$

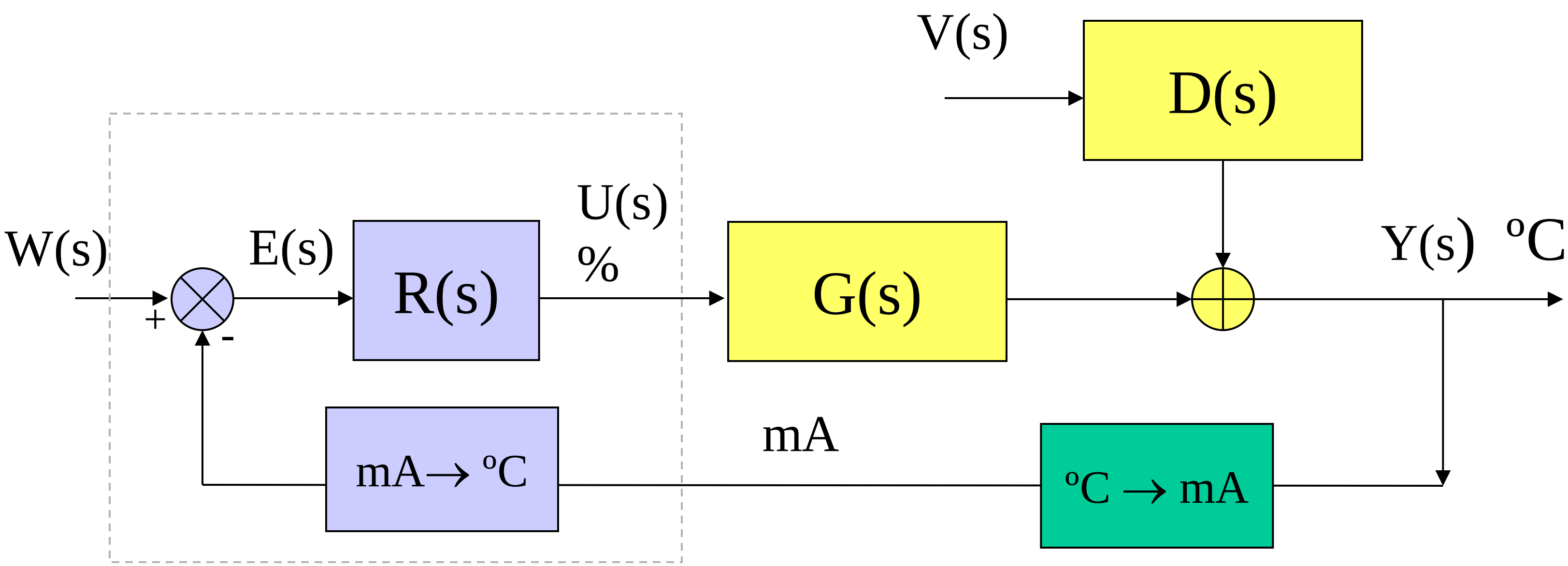
Disturbances



$$Y(s) = G(s)U(s) + D(s)V(s) = G(s)R(s)E(s) + D(s)V(s) =$$
$$= G(s)R(s)\left[W(s) - Y(s)H(s)\right] + D(s)V(s)$$
$$Y(s)\left[1 + G(s)R(s)H(s)\right] = G(s)R(s)W(s) + D(s)V(s)$$

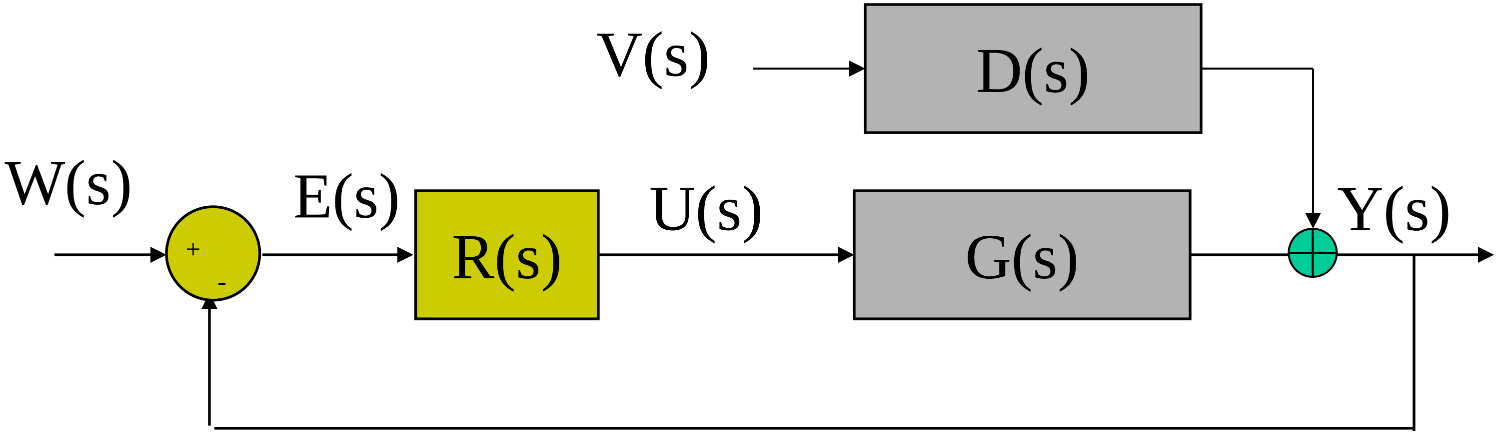
$$Y(s) = \frac{G(s)R(s)}{1 + G(s)R(s)H(s)} W(s) + \frac{D(s)}{1 + G(s)R(s)H(s)} V(s)$$

Transmitter-Controller



If the controller uses the transmitter calibration and the transmitter dynamics is fast compared with the one of the process, then the feedback dynamics can be omitted

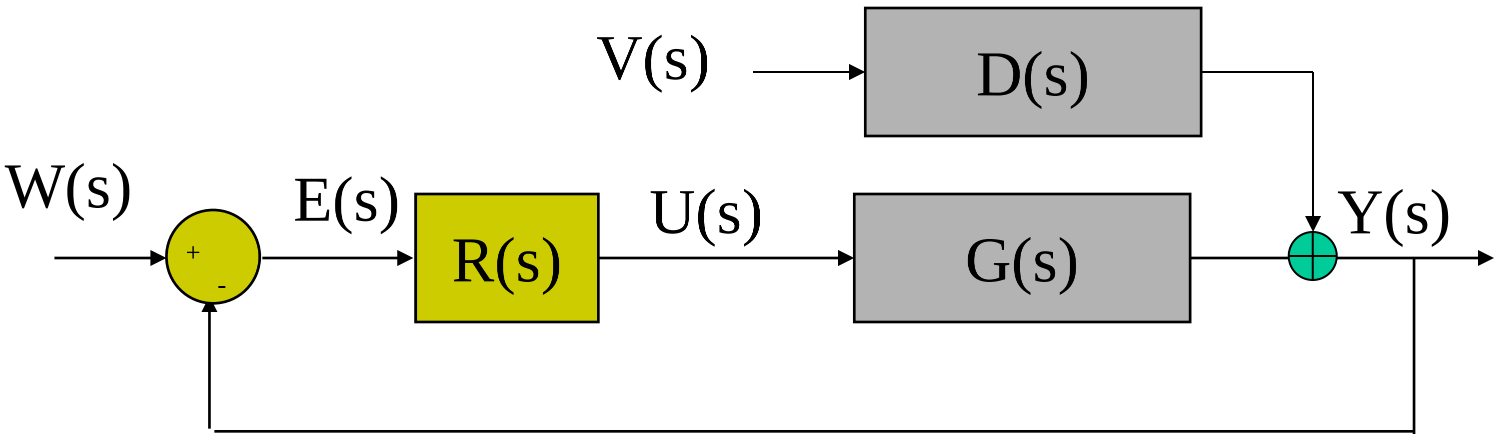
Closed loop



$$Y(s) = \frac{G(s)R(s)}{1+G(s)R(s)}W(s) + \frac{D(s)}{1+G(s)R(s)}V(s)$$

Key relation for feedback systems analysis and design

Closed loop - Control signal

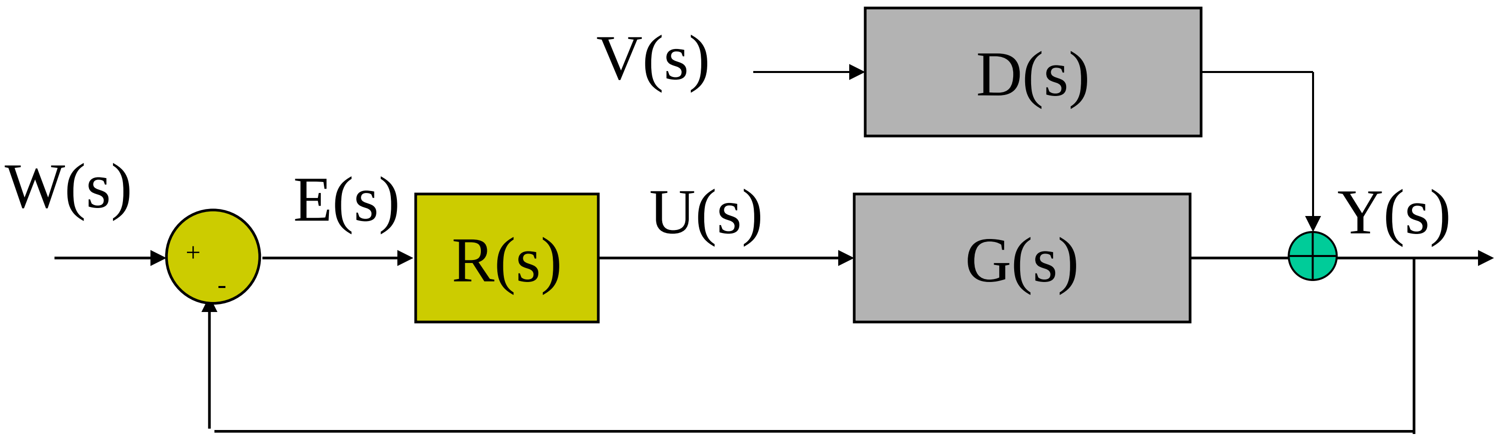


$$U(s) = R(s)E(s) = R(s)[W(s) - Y(s)] = R(s)[W(s) - G(s)U(s) - D(s)V(s)] =$$

$$U(s)[1 + R(s)G(s)] = R(s)[W(s) - D(s)V(s)]$$

$$U(s) = \frac{R(s)}{1 + G(s)R(s)} W(s) + \frac{R(s)D(s)}{1 + G(s)R(s)} V(s)$$

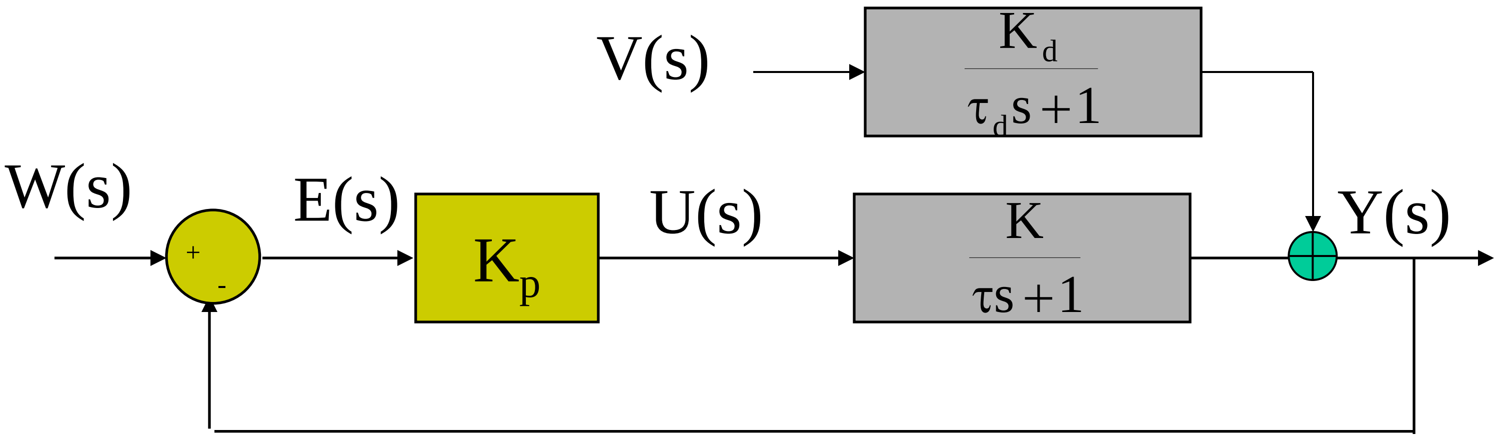
Time response in closed loop



The time response of the closed loop system under changes in w(t) or v(t) can be computed from the closed loop poles and zeros using the previous analysis

$$Y(s) = \frac{G(s)R(s)}{1+G(s)R(s)} W(s) + \frac{D(s)}{1+G(s)R(s)} V(s)$$

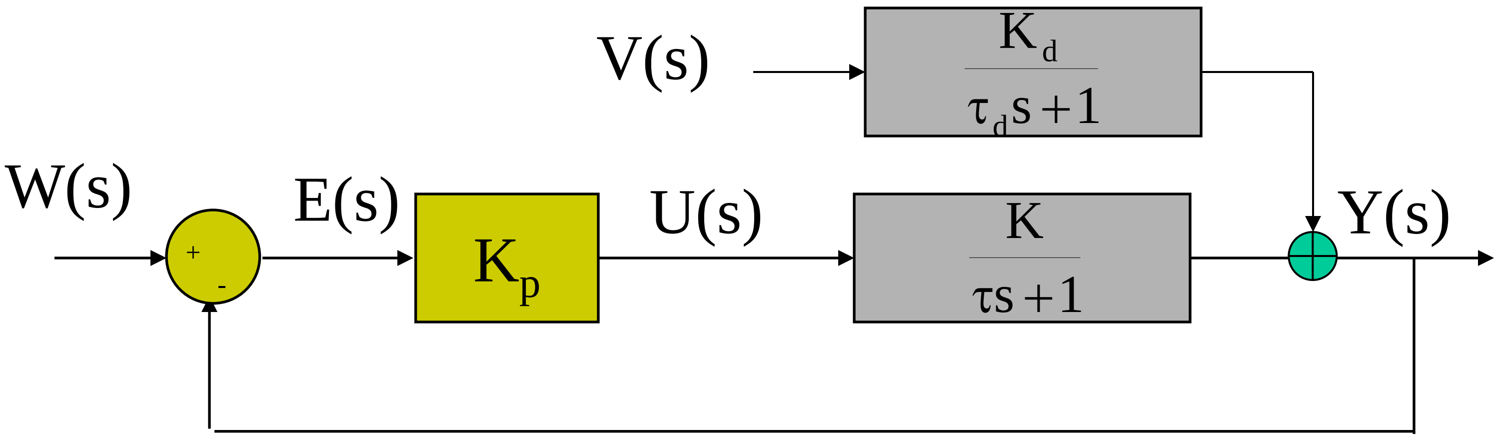
Example



$$Y(s)=\frac{G(s)K_p}{1+G(s)K_p}W(s)+\frac{D(s)}{1+G(s)K_p}V(s)=\frac{\frac{K}{\tau s+1}K_p}{1+\frac{K}{\tau s+1}K_p}W(s)+\frac{\frac{K_d}{\tau_d s+1}}{1+\frac{K}{\tau s+1}K_p}V(s)=$$

$$=\frac{KK_p}{\tau s+1+KK_p}W(s)+\frac{K_d(\tau s+1)}{(\tau s+1+KK_p)(\tau_d s+1)}V(s)$$

Example



$$Y(s) = \frac{KK_p}{\tau s + 1 + KK_p} W(s) + \frac{K_d(\tau s + 1)}{(\tau s + 1 + KK_p)(\tau_d s + 1)} V(s)$$

For positive KK_p, stable overdamped response with no change in concavity against SP step changes and with change in concavity and an advanced response if the disturbance v experiences a step change

Characteristic equation

$$Y(s) = \frac{G(s)R(s)}{1+G(s)R(s)} W(s) + \frac{D(s)}{1+G(s)R(s)} V(s)$$

The type of response and the stability in closed loop are given by the poles of the closed loop TF, which correspond to the roots of the characteristic equation:

$$1+G(s)R(s) = 0$$

Changing the controller R(s), the closed loop time response can be modified. Notice that the closed loop dynamics can be completely different from the open loop one

Closed loop zeros

$$Y(s) = \frac{G(s)R(s)}{1+G(s)R(s)} W(s) + \frac{D(s)}{1+G(s)R(s)} V(s)$$

$$G(s)R(s) = \frac{Num(s)}{Den(s)}$$

$$\frac{G(s)R(s)}{1+G(s)R(s)} = \frac{\frac{Num(s)}{Den(s)}}{1+\frac{Num(s)}{Den(s)}} = \frac{Num(s)}{Den(s)+Num(s)}$$

$$\frac{D(s)}{1+G(s)R(s)} = \frac{D(s)}{1+\frac{Num(s)}{Den(s)}} = \frac{Den(s)D(s)}{Den(s)+Num(s)}$$

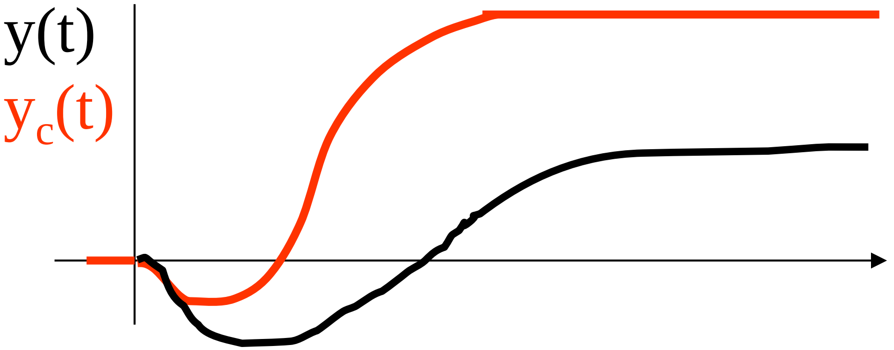
The open loop zeros appear also as zeros of the closed loop TF

Right half plane zeros (unstable zeros)

$$G(s)R(s) = \frac{Num(s)}{Den(s)}$$

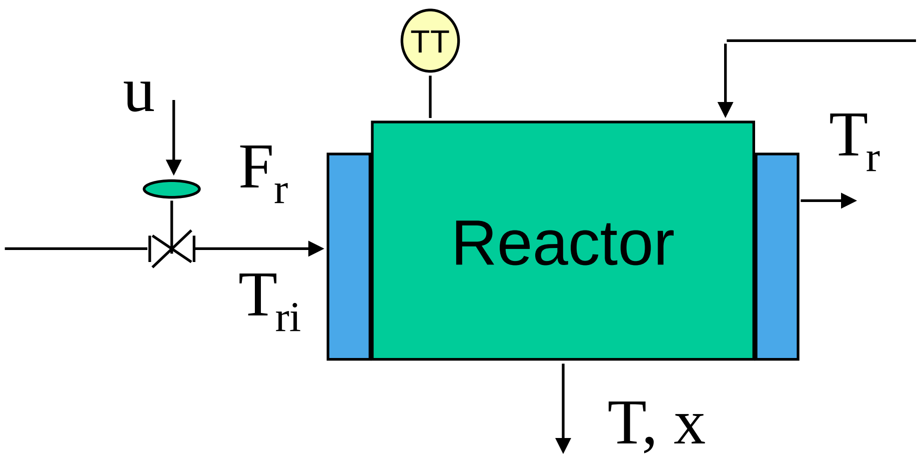
$$\frac{G(s)R(s)}{1+G(s)R(s)} = \frac{\frac{Num(s)}{Den(s)}}{1+\frac{Num(s)}{Den(s)}} = \frac{Num(s)}{Den(s)+Num(s)}$$

$$\frac{D(s)}{1+G(s)R(s)} = \frac{D(s)}{1+\frac{Num(s)}{Den(s)}} = \frac{Den(s)D(s)}{Den(s)+Num(s)}$$



If the open loop time response is of minimum phase type, the closed loop time response will be similar, independently of the controller R(s)

Chemical reactor



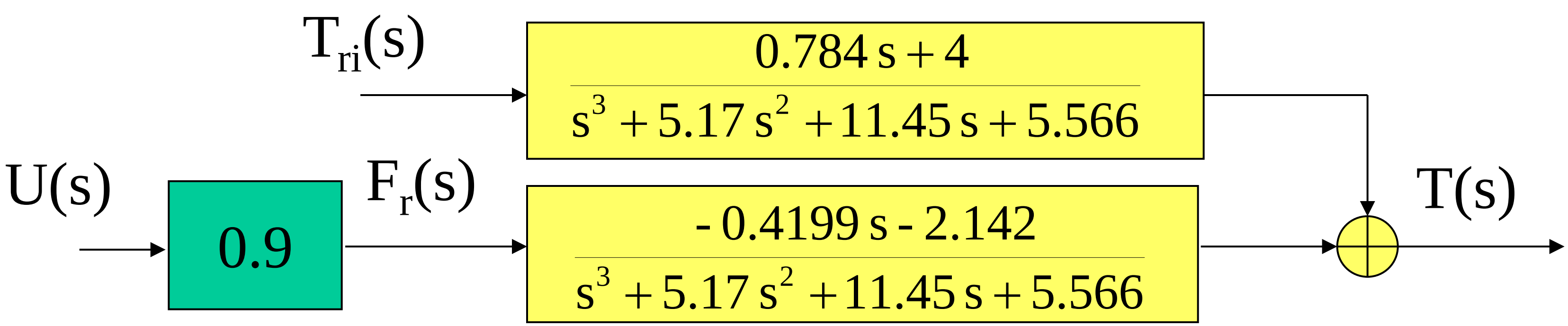
At the operating point:

$T = 92\ ^{o}C \quad x = 0.902$

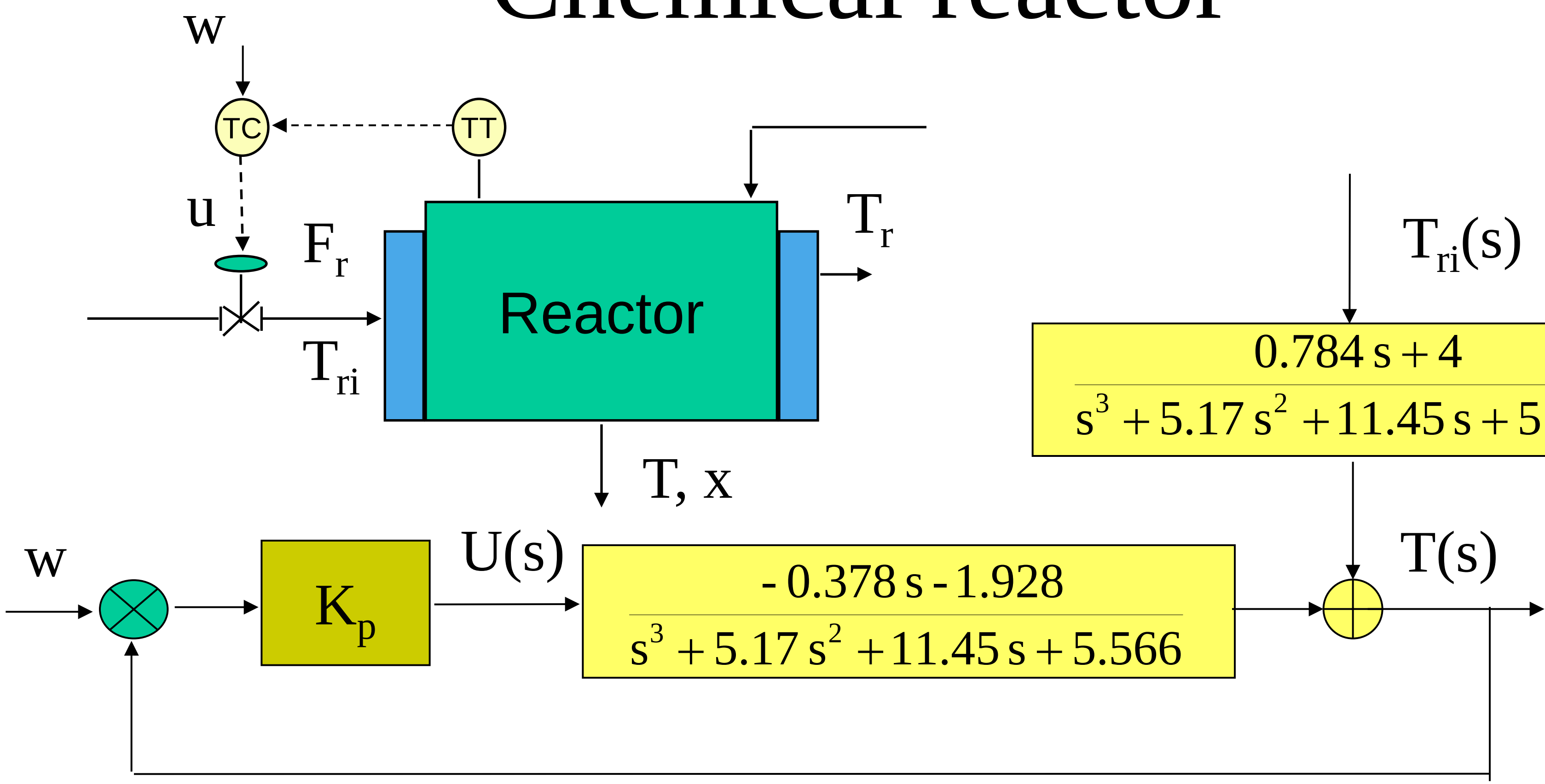
$T_r = 75.6\ ^{o}C$

$F_r = 47.8\ l/m$

$T_{ri} = 50\ ^{o}C \quad u = 42\ \%$



Chemical reactor



Chemical reactor

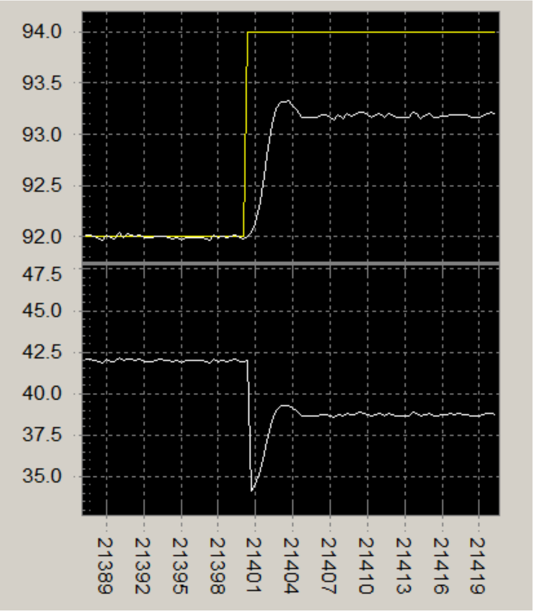
1+G(s)R(s) = 0

$$1+\frac{K_p(-0.378\,s-1.928)}{s^3+5.17\,s^2+11.45\,s+5.566}=0$$

$$s^3+5.17\,s^2+11.45\,s+5.566+K_p(-0.378\,s-1.928)=0$$

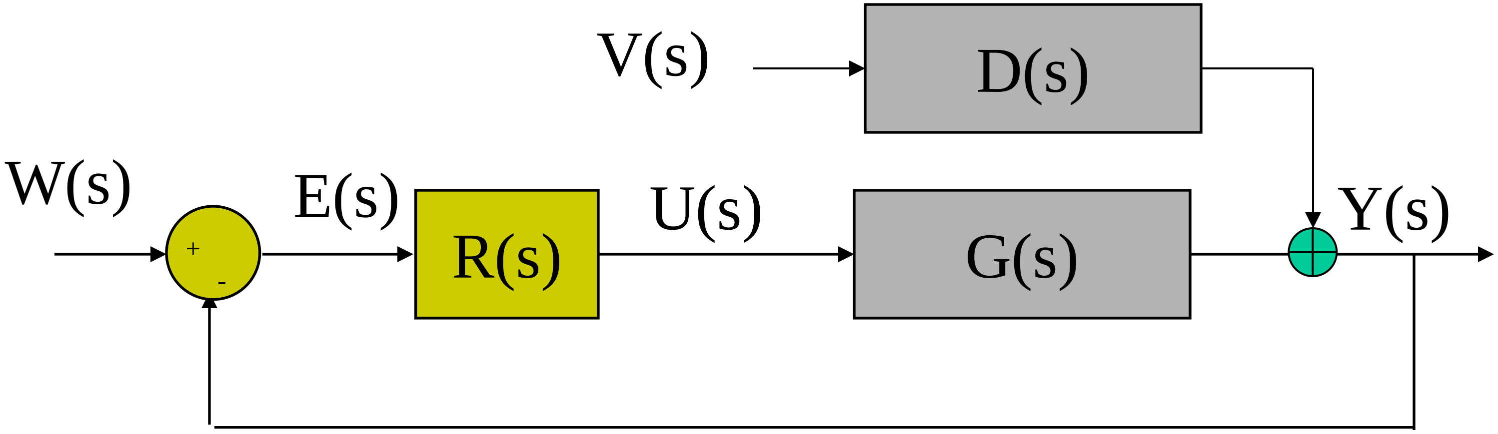
For K_p = - 4 the closed loop poles are:
-1.5810 + 2.0281i
-1.5810 - 2.0281i
-2.0079

Also a zero at: -5.1



Step response to a change of 2 degrees in the SP

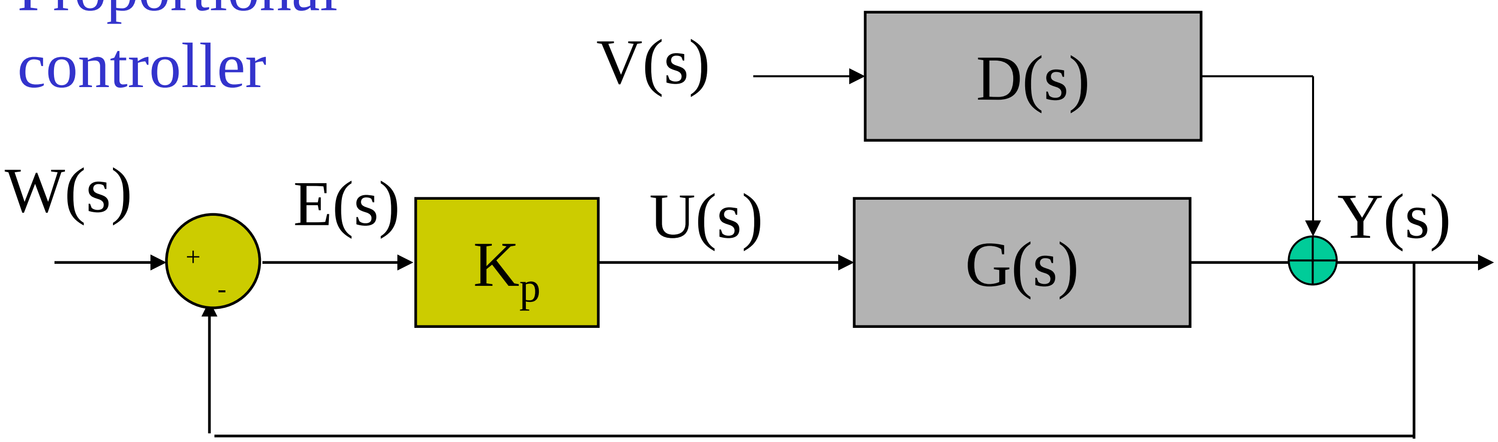
Closed loop



$$Y(s) = \frac{G(s)R(s)}{1+G(s)R(s)} W(s) + \frac{D(s)}{1+G(s)R(s)} V(s)$$

Changes of the closed loop dynamics as functions of changes in the controller parameters

Proportional controller



$$Y(s) = \frac{G(s)K_p}{1+G(s)K_p} W(s) + \frac{D(s)}{1+G(s)K_p} V(s)$$

$$\text{Ecuación característica : } \; 1+K_p G(s) = 0$$

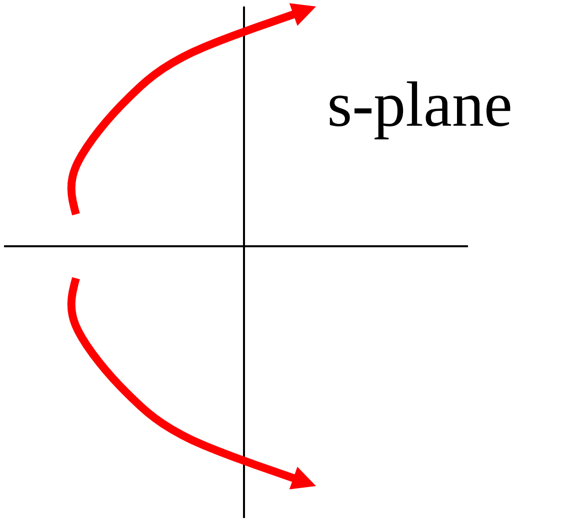
Root locus

$$Y(s) = \frac{G(s)K_p}{1+G(s)K_p} W(s) + \frac{D(s)}{1+G(s)K_p} V(s)$$

Characteristic equation: $1 + K_p G(s) = 0$

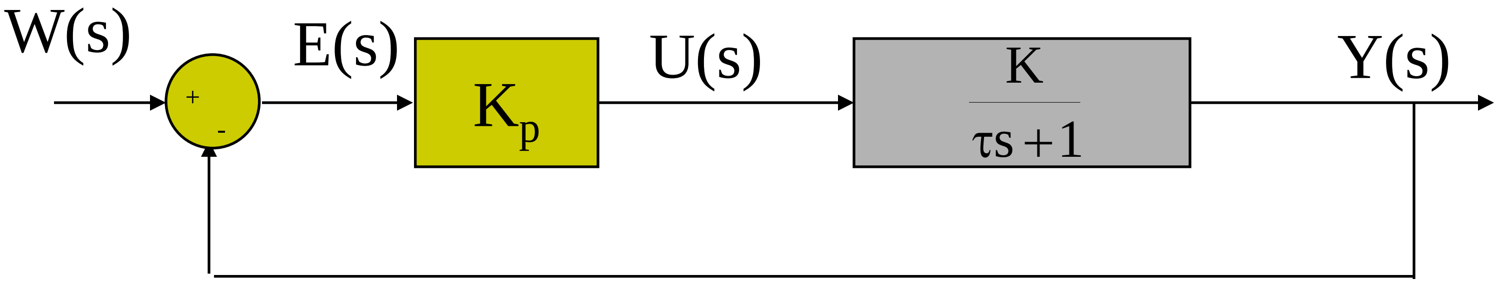
The root locus is a representation in the s-plane of the closed loop poles for different values of the controller gain K_p (and eventually any other paramenter)

It allows to know the closed loop stability and the types of dynamic response that corresponds to different values of the controller gain.



The root locus must be symmetric respect to the real axis

First order systems



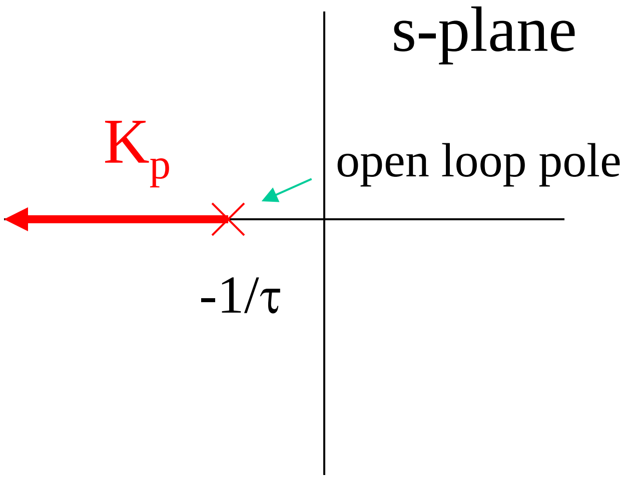
Characteristic equation: $1 + K_p G(s) = 0$

$$1 + K_p \frac{K}{\tau s + 1} = 0 \qquad \tau s + 1 + K_p K = 0$$

$$s = -\frac{1 + K_p K}{\tau}$$

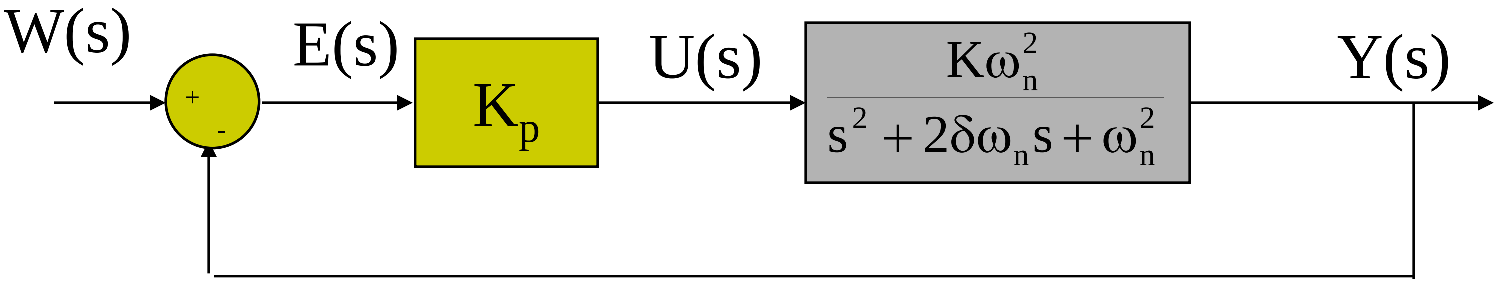
Overdamped response with decreasing settling time for increasing K_p

Faster response in closed than in open loop



The root locus stars in the open loop pole.

Second order systems



$$\text{Characteristic equation: } \quad 1 + K_p G(s) = 0$$

$$1 + K_p \frac{K\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2} = 0 \qquad s^2 + 2\delta\omega_n s + \omega_n^2 + K_p K\omega_n^2 = 0$$

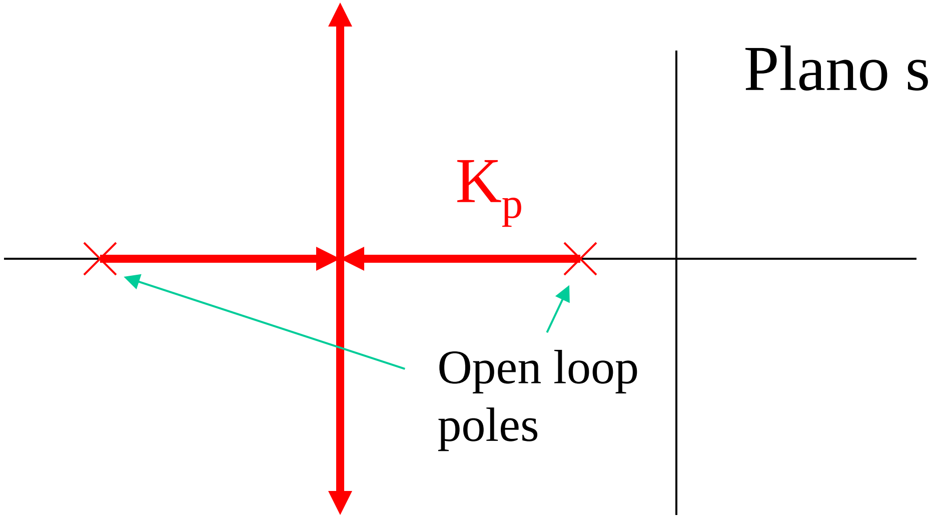
$$s = \frac{-2\delta\omega_n \pm \sqrt{4\delta^2\omega_n^2 - 4(\omega_n^2 + K_p K\omega_n^2)}}{2} =$$

$$s = -\delta\omega_n \pm \omega_n \sqrt{\delta^2 - 1 - K_p K}$$

Second order systems

If the open loop process is overdamped, then, when K_p is increased from zero, the closed loop response is initially also overdamped and increasingly faster, but, above a certain gain, the response becomes underdamped with constant settling time and increasing overshoot and oscillation frequency

$$s = -\delta\omega_n \pm \omega_n \sqrt{\delta^2 - 1 - K_p K}$$

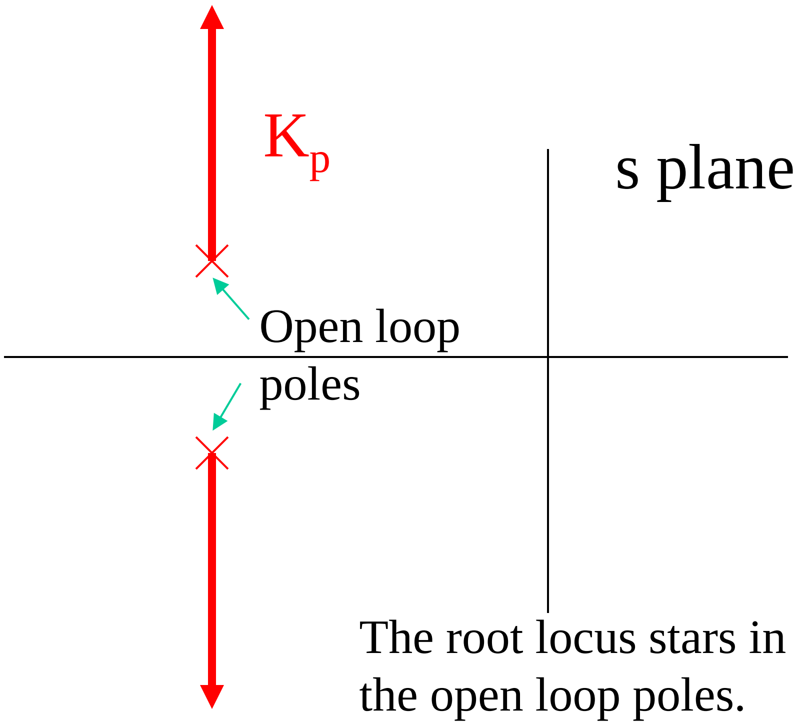


The root locus stars in the open loop poles.

Second order systems

$$s = -\delta\omega_n \pm \omega_n\sqrt{\delta^2 - 1 - K_p K}$$

If the open loop process is underdamped, then, when K_p is increased from zero, the closed loop response is also underdamped with constant settling time and increasing overshoot and oscillation frequency



The root locus stars in the open loop poles.

Root locus

$$1 + K_p G(s) = 1 + K_p \frac{Num(s)}{Den(s)} = 0$$

$$Den(s) + K_p Num(s) = 0$$

$$\text{for } K_p = 0 \quad \Rightarrow \quad Den(s) = 0$$

the root locus starts in the open loop poles

$$\text{for } K_p = \infty \quad \Rightarrow \quad Num(s) = 0$$

the root locus ends at the open loop zeros

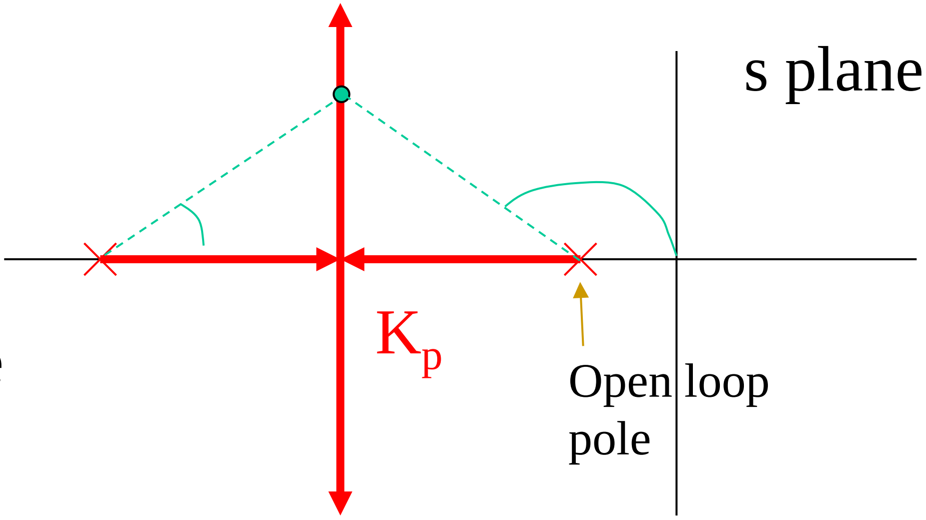
Extra zeros located at infinite can be considered to exist (up to equating the number of poles and zeros)

$$\frac{K(\tau_3 s + 1)}{(\tau_1 s + 1)(\tau_1 s + 1)} \qquad \frac{(\frac{1}{\infty} s + 1) K(\tau_3 s + 1)}{(\tau_1 s + 1)(\tau_1 s + 1)}$$

Root locus

$$1 + K_p G(s) = 0$$

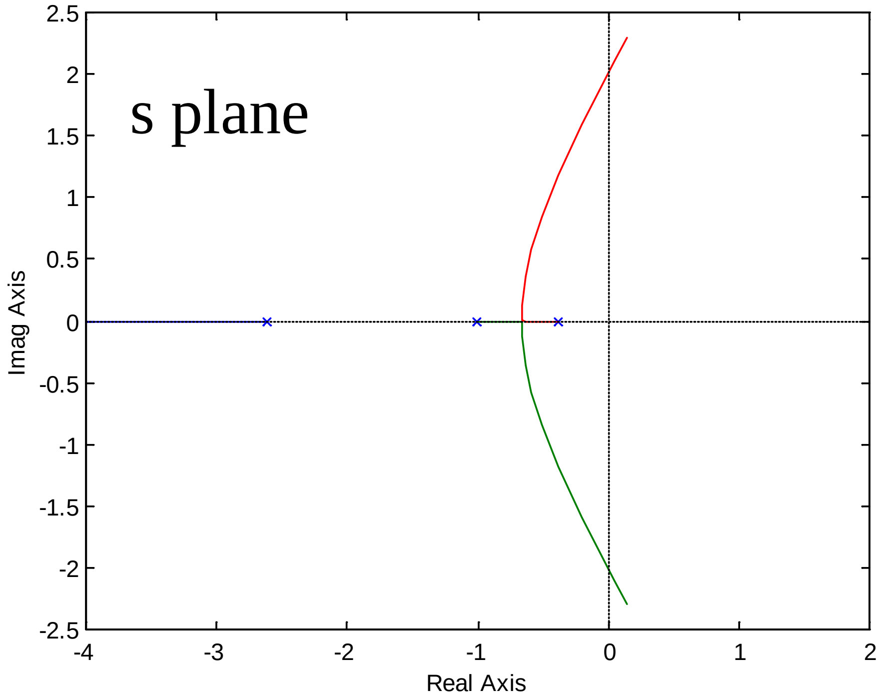
$$G(s) = \frac{-1}{K_p}$$



For any point s on the root locus, G(s) has argument $-\pi$

Sisotool

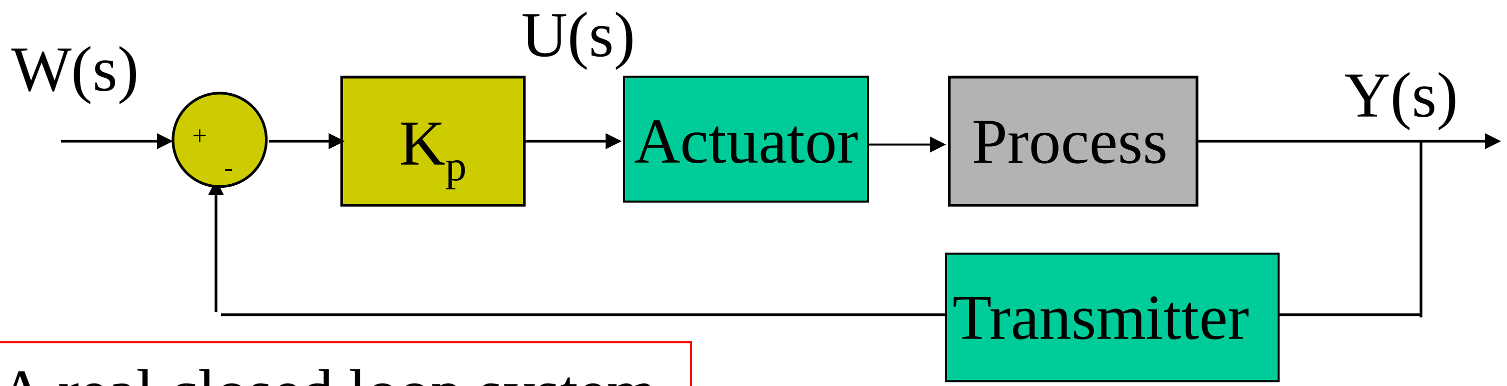
Third order systems



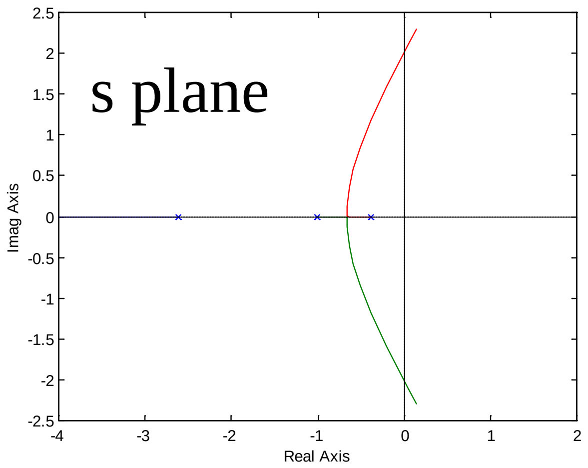
$$\frac{1}{s^3 + 4\,s^2 + 4\,s + 1}$$

With increasing K_p, the system response is more oscillatory and can become unstable

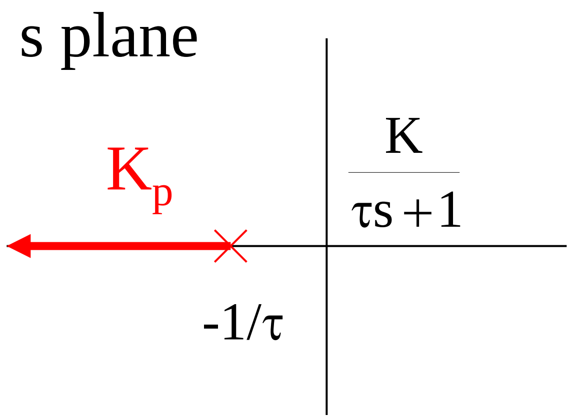
Real closed loop systems



A real closed loop system is always of third or higher order due to the dynamics of actuators and transmitters. Accordingly, a high value of K_p will tend to destabilize the closed loop system.

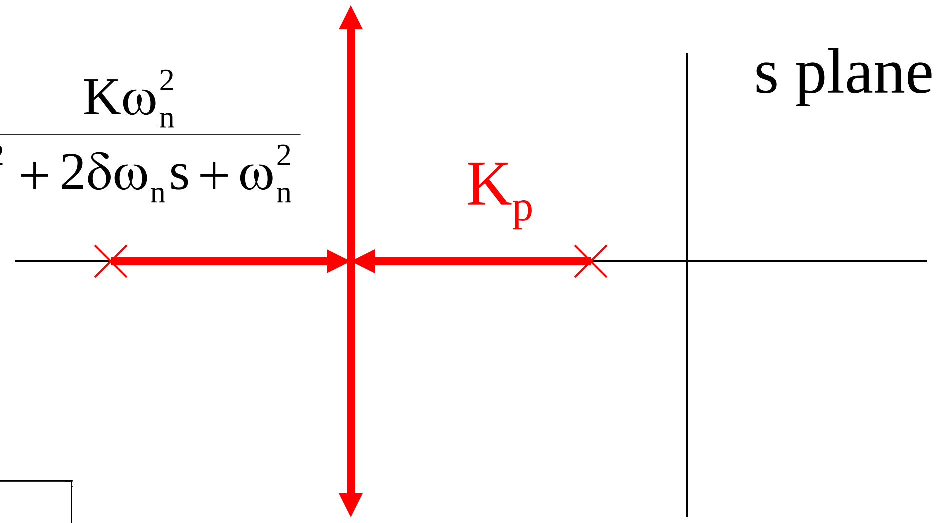


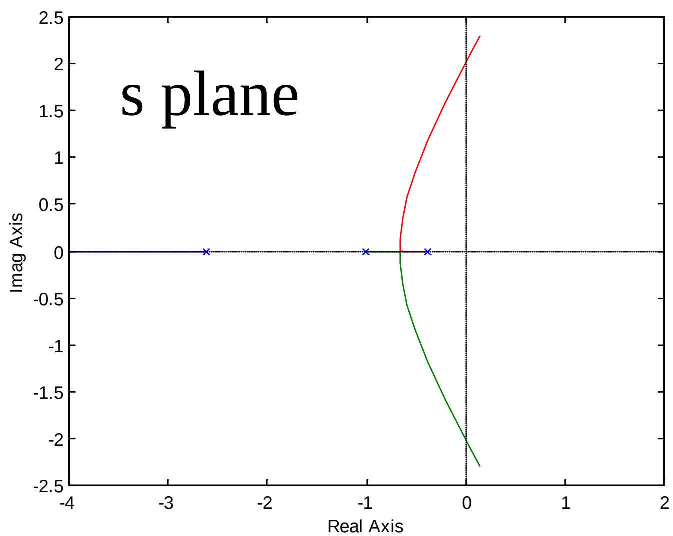
Root locus



$$\frac{K}{\tau s+1}$$

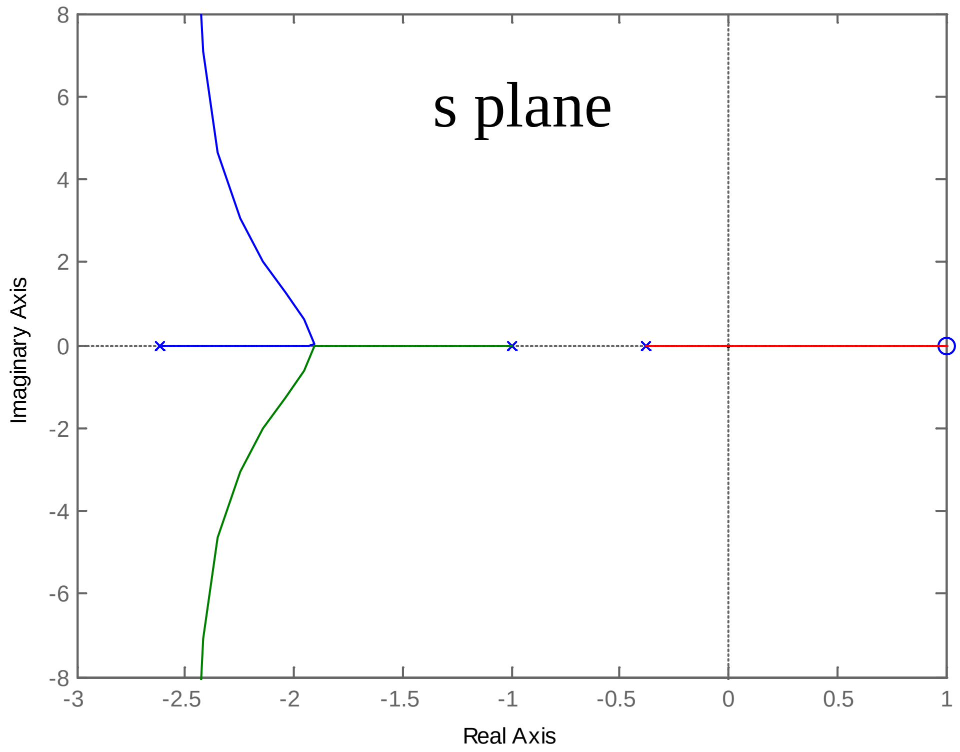
$$\frac{K\omega_n^2}{s^2+2\delta\omega_n s+\omega_n^2}$$





$$\frac{K}{(\tau_1 s+1)(\tau_2 s+1)(\tau_3 s+1)}$$

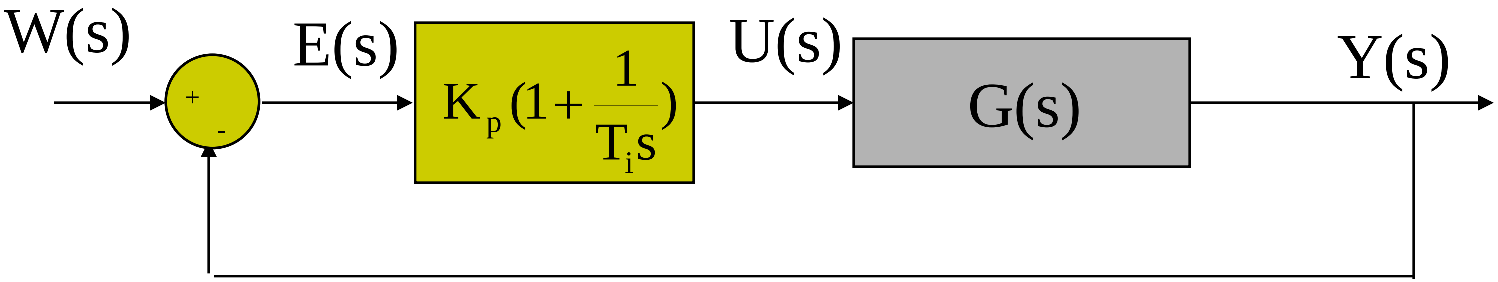
Zeros in the right hand side



$$\frac{s-1}{s^3 + 4\,s^2 + 4\,s + 1}$$

As the root locus ends at the open loop zeros, if there are unstable zeros in open loop, then the closed loop system will became unstable for increasing values of K_p

PI+G(s)

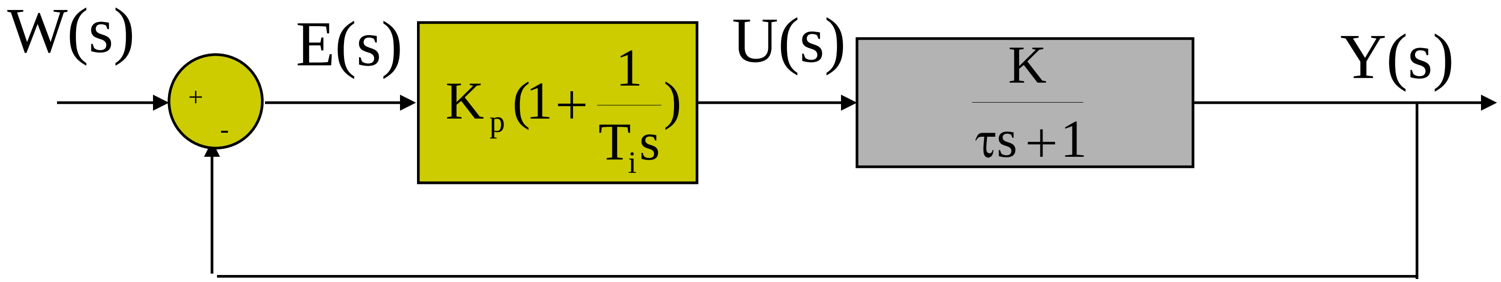


$$\text{Characteristic equation: } 1 + R(s)G(s) = 0$$

$$1 + K_p \frac{T_i s + 1}{T_i s} G(s) = 0$$

For a given T_i one can draw the root locus of the "extended" system $(T_i s+1)G(s)/s$

PI + First order



$$\text{Characteristic equation: } 1+R(s)G(s)=0$$

$$1+K_p\frac{T_i s+1}{T_i s}\frac{K}{\tau s+1}=0 \qquad T_i s(\tau s+1)+K_pK(T_i s+1)=0$$

$$T_i\tau s^2+T_i(1+K_pK)s+K_pK=0$$

$$s=\frac{-T_i(1+K_pK)\pm\sqrt{T_i^2(1+K_pK)^2-4T_i\tau K_pK}}{2T_i\tau}$$

$$s=\frac{-(1+K_pK)\pm\sqrt{(1+K_pK)^2-4\tau K_pK/T_i}}{2\tau}$$

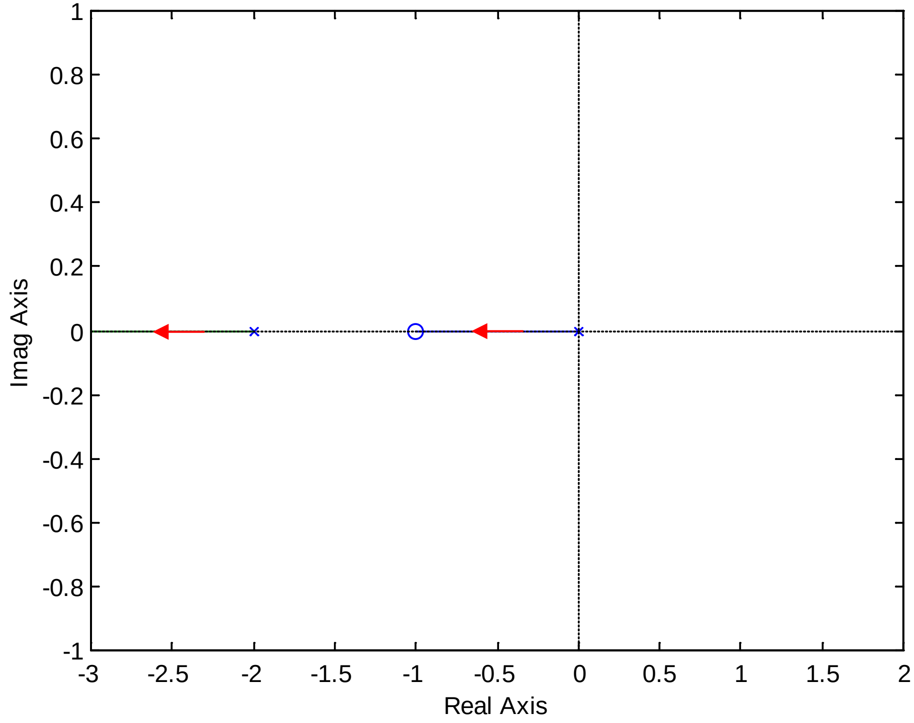
The root locus can be drawn for any given T_i

PI + First order

$$\frac{K}{\tau s + 1} = \frac{1}{0.5s + 1}$$

$$K_p(1 + \frac{1}{T_i s}) = K_p(1 + \frac{1}{s})$$

$$T_i = 1 \; , \;\; \tau = 0.5$$

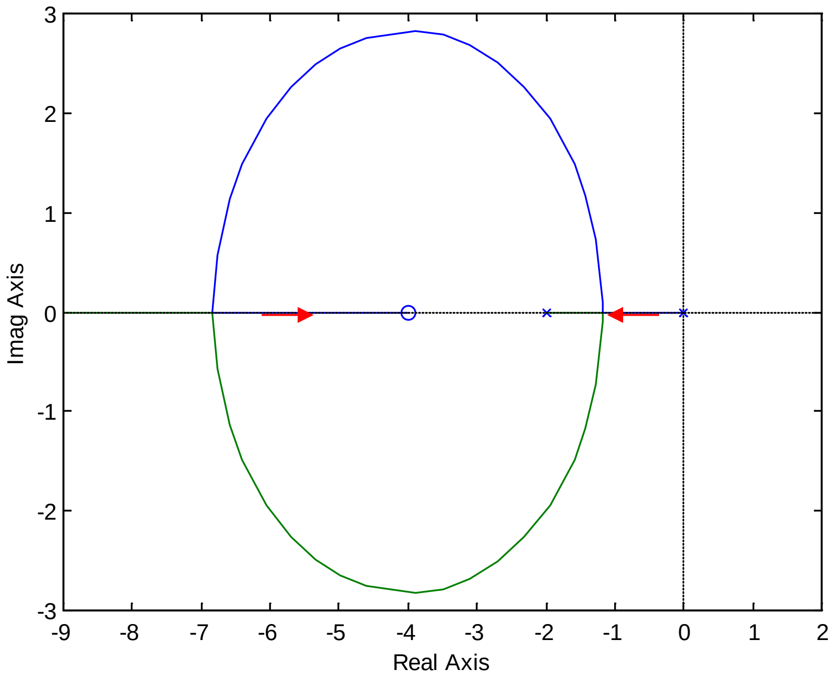


PI + First order

$$\frac{K}{\tau s+1}=\frac{1}{0.5s+1}$$

$$K_p(1+\frac{1}{T_i s})=K_p(1+\frac{1}{0.25s})$$

$$T_i = 0.25\ ,\ \ \tau = 0.5$$



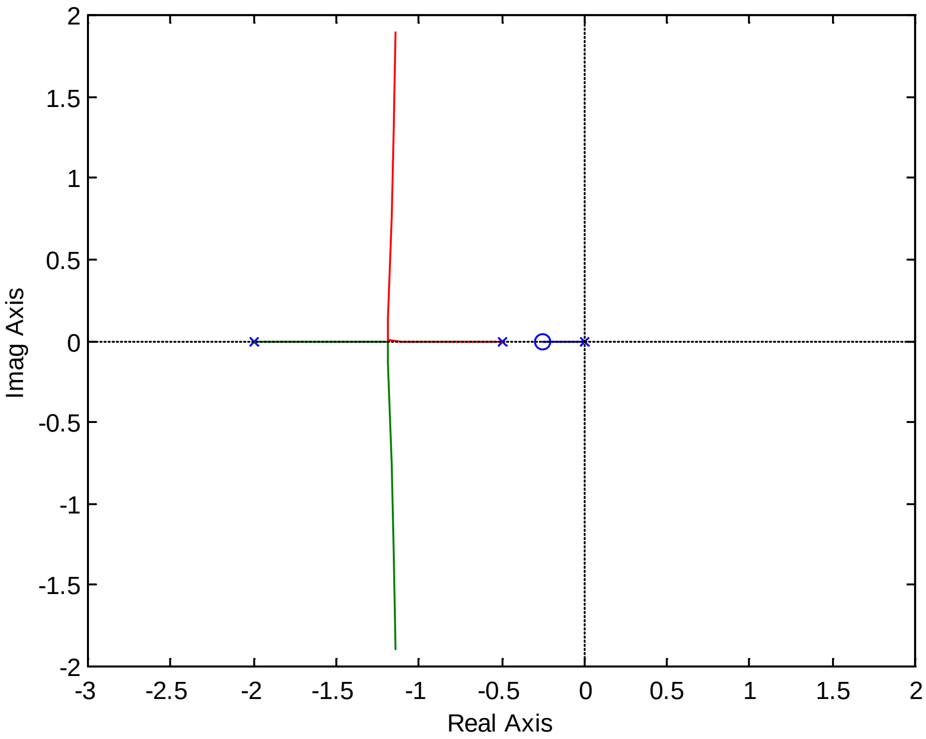
SysQuake

PI + G(s)

$$G(s) = \frac{1}{(0.5s+1)(2s+1)}$$

$$K_p(1+\frac{1}{T_i s}) = K_p(1+\frac{1}{4s})$$

$$T_i = 4$$

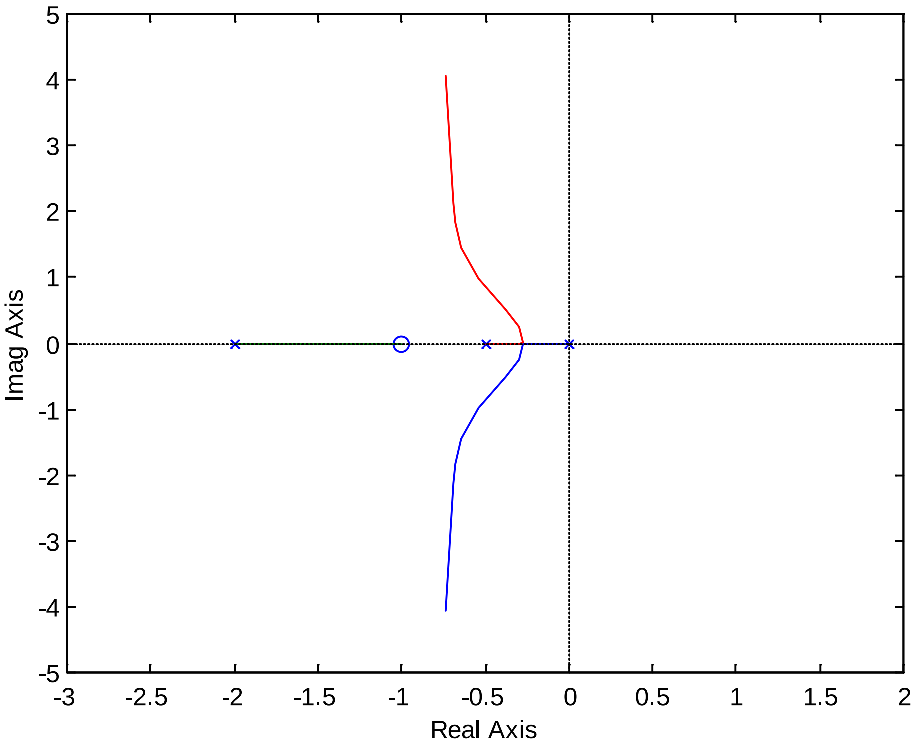


PI + G(s)

$$G(s) = \frac{1}{(0.5s+1)(2s+1)}$$

$$K_p(1+\frac{1}{T_i s}) = K_p(1+\frac{1}{s})$$

$$T_i = 1$$

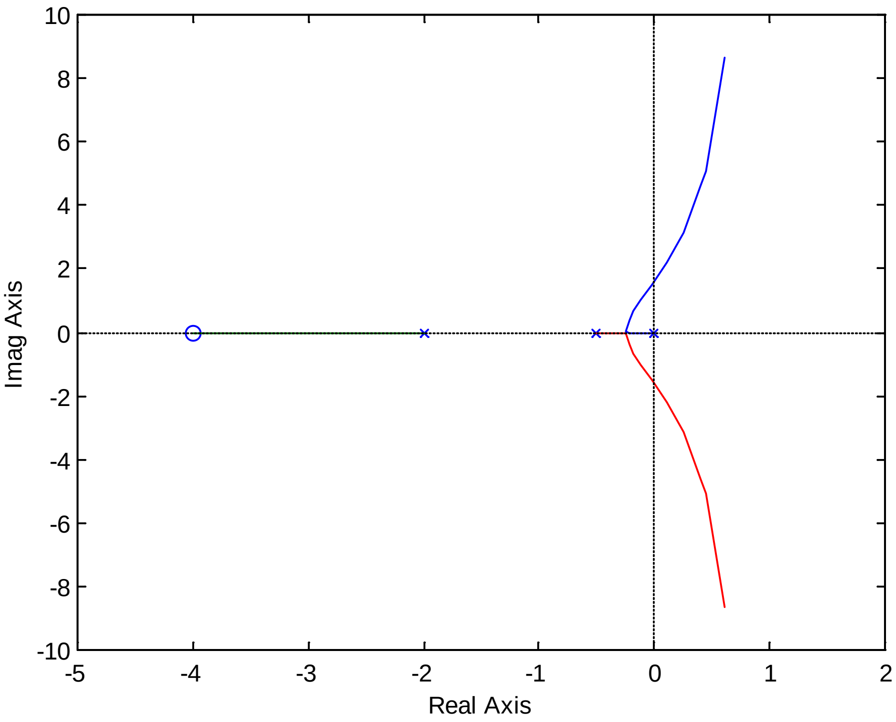


PI + G(s)

$$G(s) = \frac{1}{(0.5s+1)(2s+1)}$$

$$K_p(1+\frac{1}{T_i s}) = K_p(1+\frac{1}{0.25s})$$

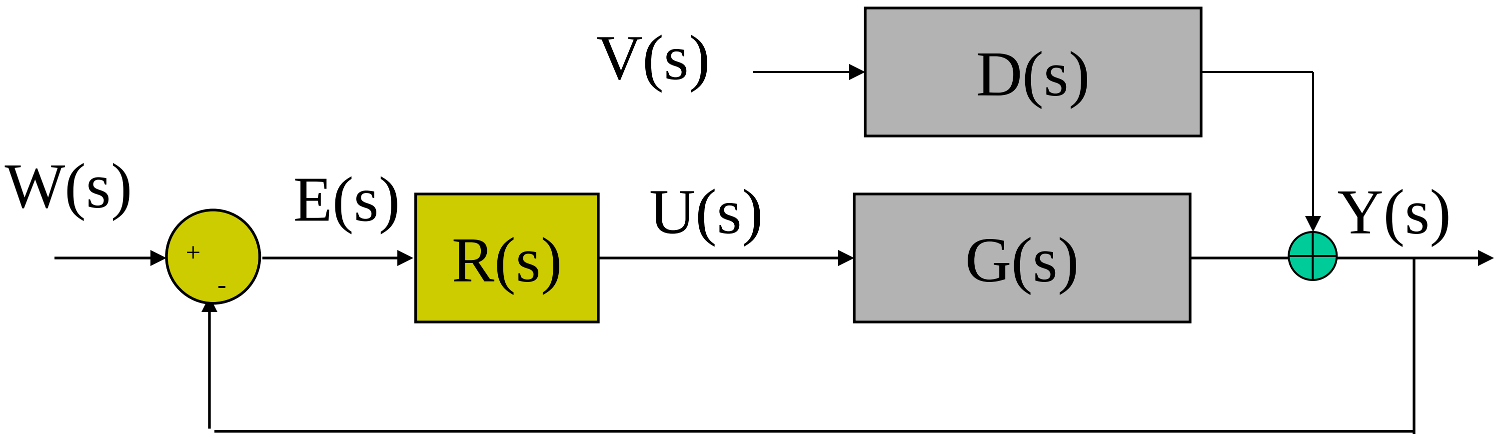
$$T_i = 0.25$$



The closed loop dynamics can vary a lot according to the relative zero position

SysQuake

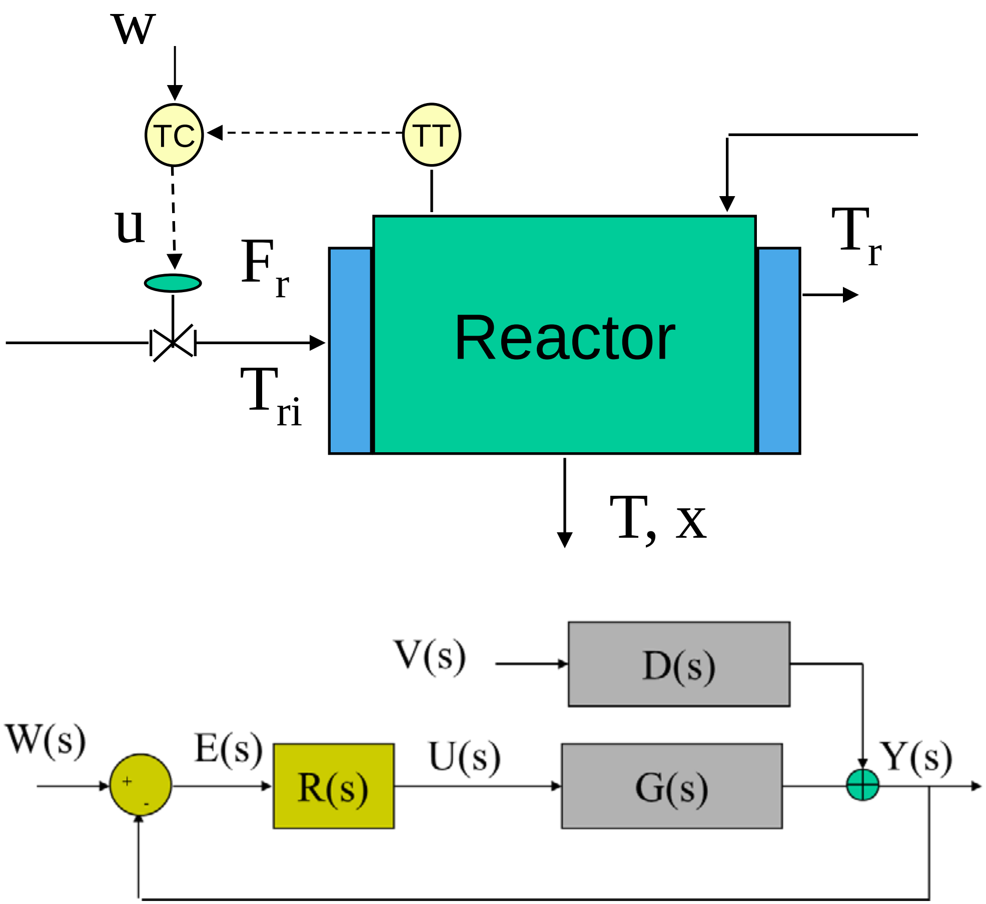
Steady state error

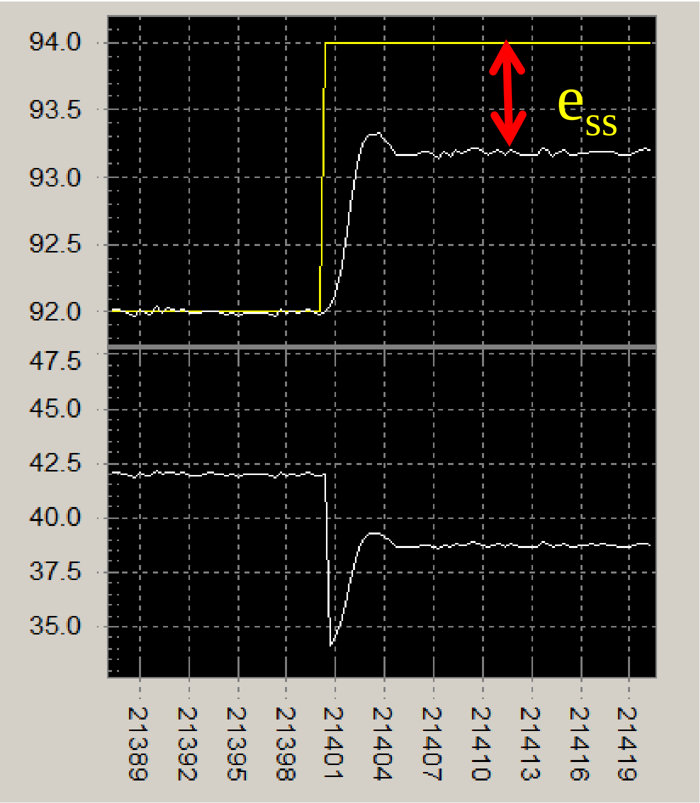


If the value of the set point changes or a disturbance appears, which will be the value of the error e(t) at steady state?

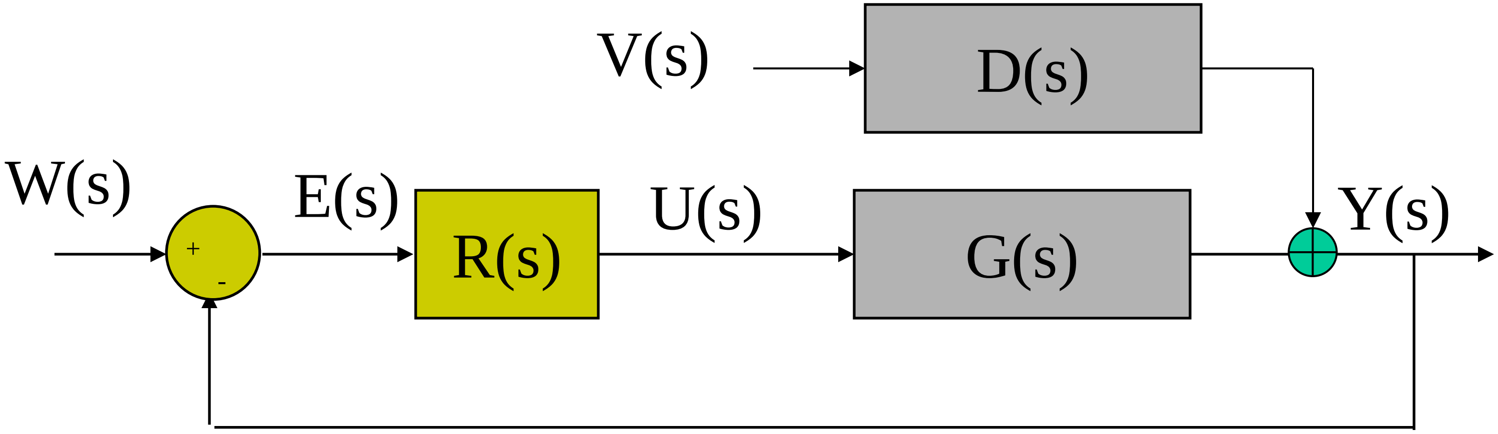
$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s)$$

Steady state error, e_{ss}





Steady state error



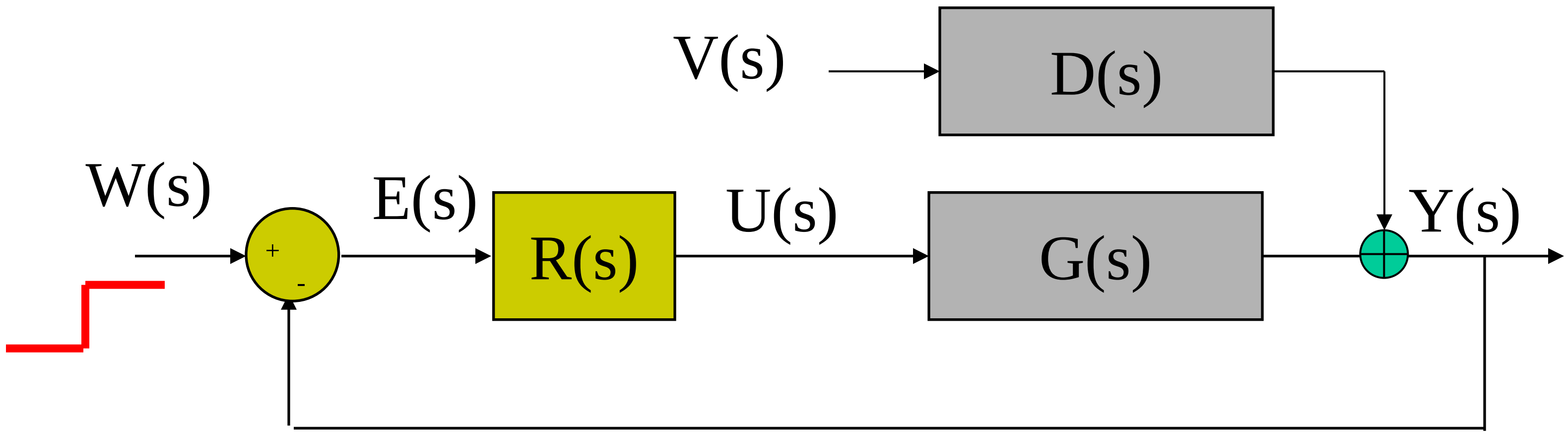
$$E(s) = W(s) - Y(s) = W(s) - [G(s)U(s) + D(s)V(s)] =$$

$$= W(s) - [G(s)R(s)E(s) + D(s)V(s)]$$

$$E(s)[1 + G(s)R(s)] = W(s) - D(s)V(s)$$

$$E(s) = \frac{1}{1 + G(s)R(s)} W(s) - \frac{D(s)}{1 + G(s)R(s)} V(s)$$

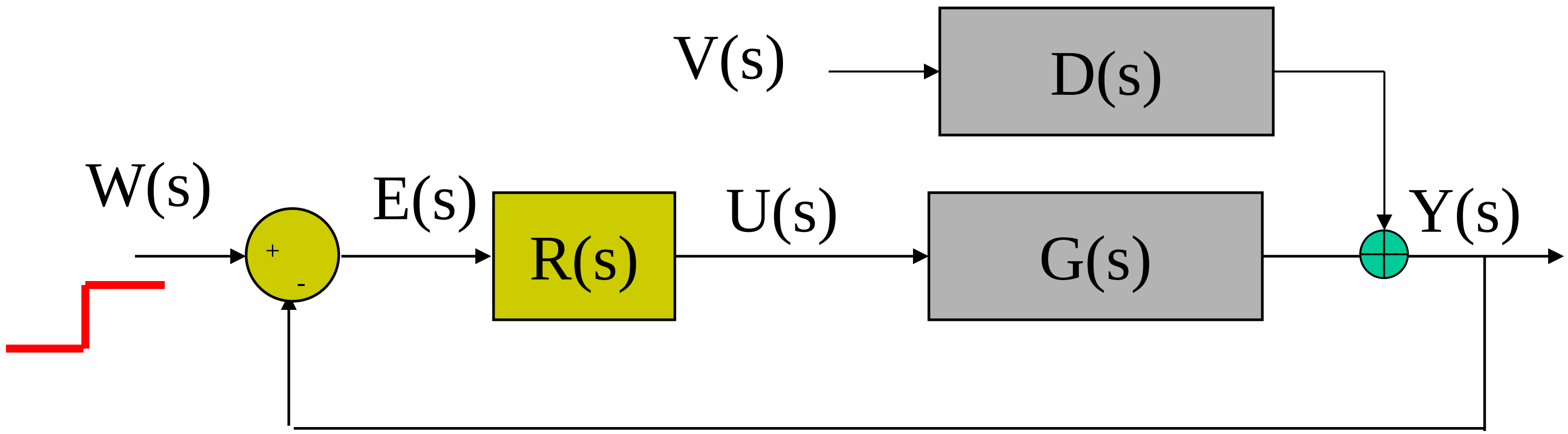
Steady state error, step on W



$$E(s) = \frac{1}{1+G(s)R(s)} W(s) - \frac{D(s)}{1+G(s)R(s)} V(s)$$

$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} s \frac{1}{1+G(s)R(s)} \frac{w}{s} = \frac{w}{1+G(0)R(0)}$$

Steady state error, step on W



$$e_{ss} = \frac{w}{1+G(0)R(0)}$$

If G(s) or R(s) have an integrator:

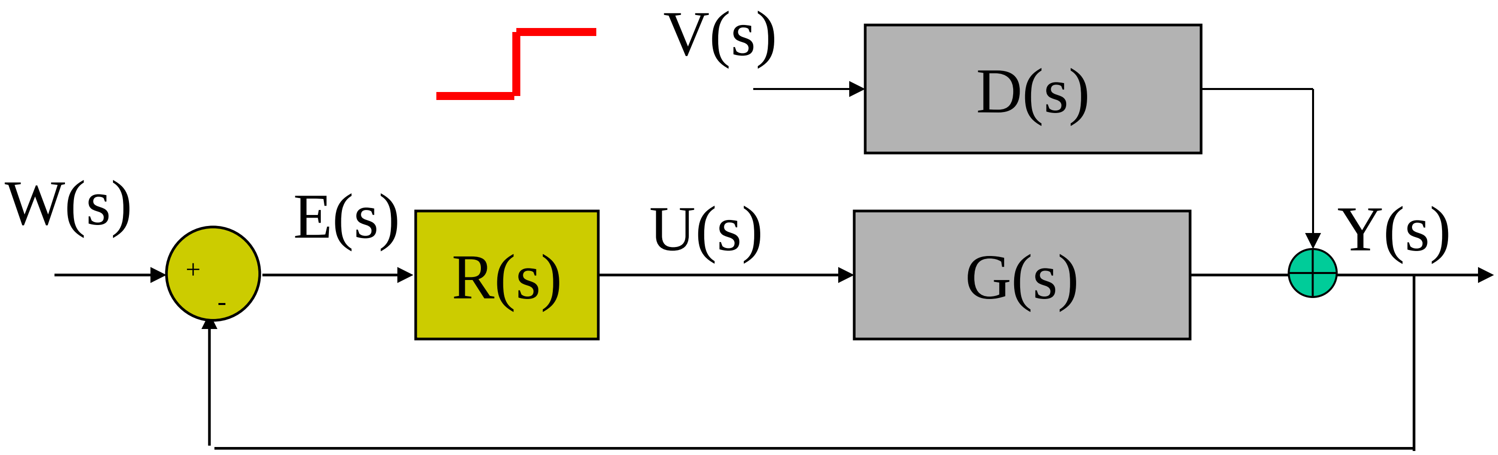
$$\frac{K(as+1)(....)}{s(bs+1)(cs+1)}$$

$$G(0)R(0) \rightarrow \infty \qquad e_{ss} = \frac{w}{1+G(0)R(0)} \rightarrow 0$$

If not, the steady state error will have a finite value, decreasing with K_p

CStation

Steady state error, step onV

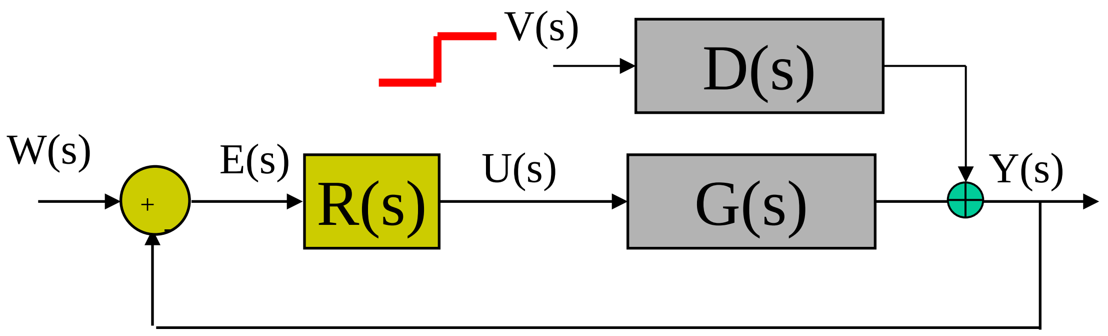


$$E(s) = \frac{1}{1+G(s)R(s)} W(s) - \frac{D(s)}{1+G(s)R(s)} V(s)$$

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s \frac{-D(s)}{1+G(s)R(s)} \frac{v}{s} = \frac{-D(0)v}{1+G(0)R(0)}$$

If D(s) have a zero at s = 0 $e_{ss} \to 0$

Steady state error, step on V



$$e_{ss} = \frac{-D(0)v}{1+G(0)R(0)}$$

If D(s) has no integrators: if G(s) or R(s) have one ntegrator:

$$\frac{K(as+1)(....)}{s(bs+1)(cs+1)}$$

$$G(0)R(0) \to \infty \qquad e_{ss} = \frac{-D(0)v}{1+G(0)R(0)} \to 0$$

If not, the steady state error will have a finite value, decreasing with Kp

Steady state error, step on V

$$e_{ss} = \frac{-D(0)v}{1+G(0)R(0)}$$

If D(s) has one integrator:

If G(s) or R(s) have one integrator:

$$G(s)R(s) = \frac{\overline{GR}(s)}{s} \qquad D(s) = \frac{\overline{D}(s)}{s} \qquad \frac{-D(s)v}{1+G(s)R(s)} = \frac{-\overline{D}(s)v}{s+\overline{GR}(s)}$$

$$e_{ss} = \frac{-\overline{D}(0)v}{\overline{GR}(0)}$$

The error will be finite

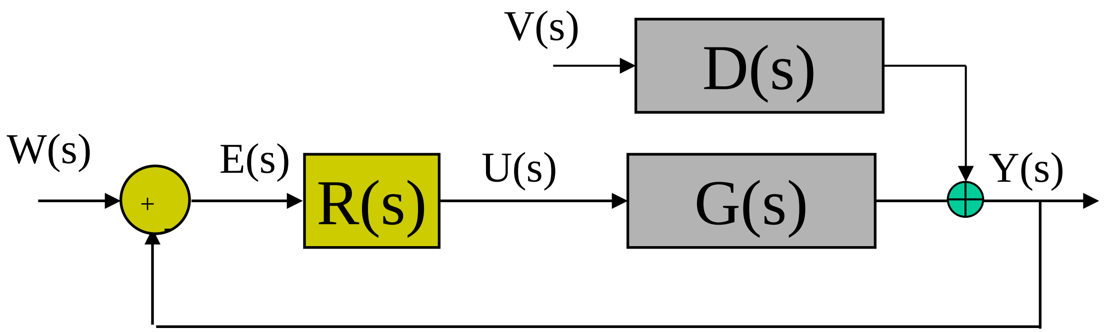
If neither G(s) nor R(s) have one integrator:

$$D(s) = \frac{\overline{D}(s)}{s} \qquad \frac{-D(s)v}{1+G(s)R(s)} = \frac{-\overline{D}(s)v}{s+sG(s)R(s)}$$

$$\frac{-\overline{D}(0)v}{0+0G(0)R(0)} \to \infty$$

Increasing error

Delays



$$e_{ss} = \frac{w}{1+G(0)R(0)}$$

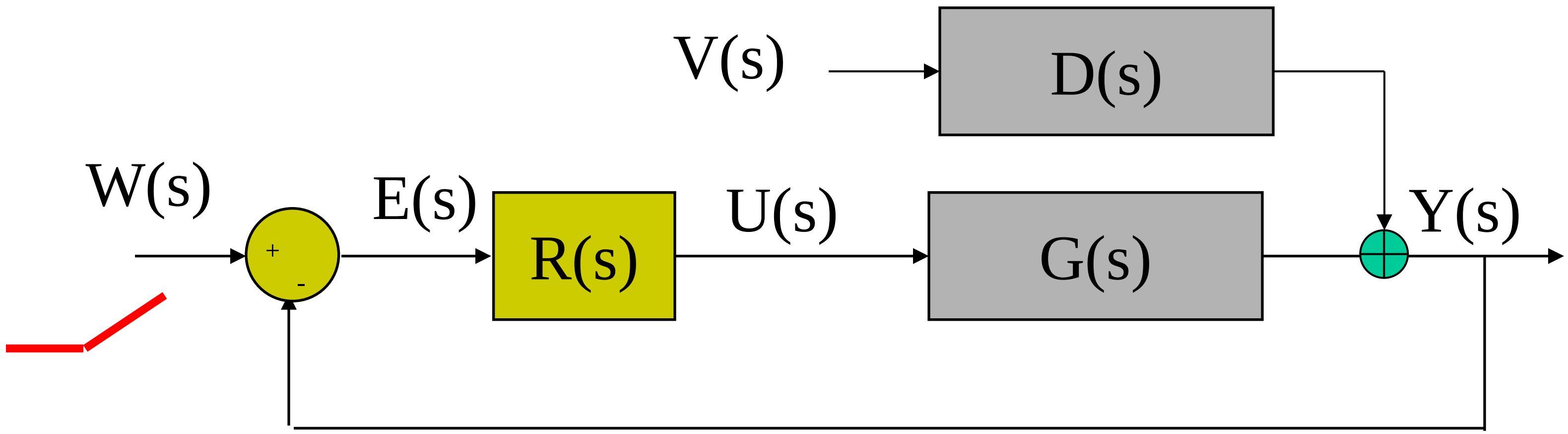
$$e_{ss} = \frac{-D(0)v}{1+G(0)R(0)}$$

The existance of delays in G(s) or D(s) does not influence the analysis of the error in steady state

$$\frac{Ke^{-ds}(as+1)(....)}{s(bs+1)(cs+1)}$$

SysQuake

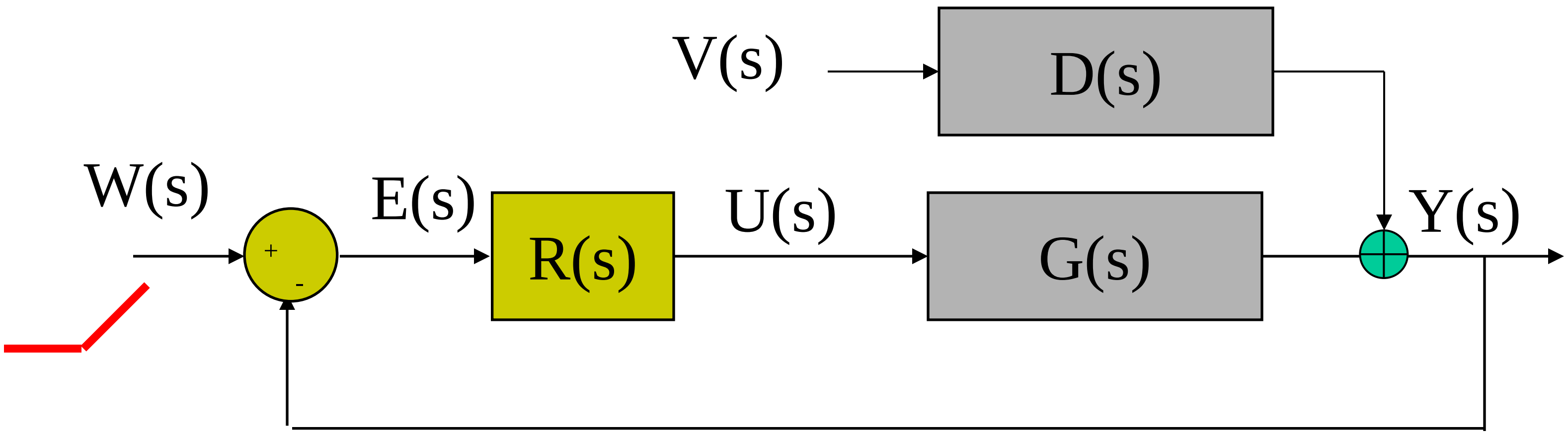
Steady state error, ramp on W



$$E(s) = \frac{1}{1+G(s)R(s)} W(s) - \frac{D(s)}{1+G(s)R(s)} V(s)$$

$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} s \frac{1}{1+G(s)R(s)} \frac{w}{s^2} = \frac{w}{sG(0)R(0)}$$

Steady state error, ramp on W



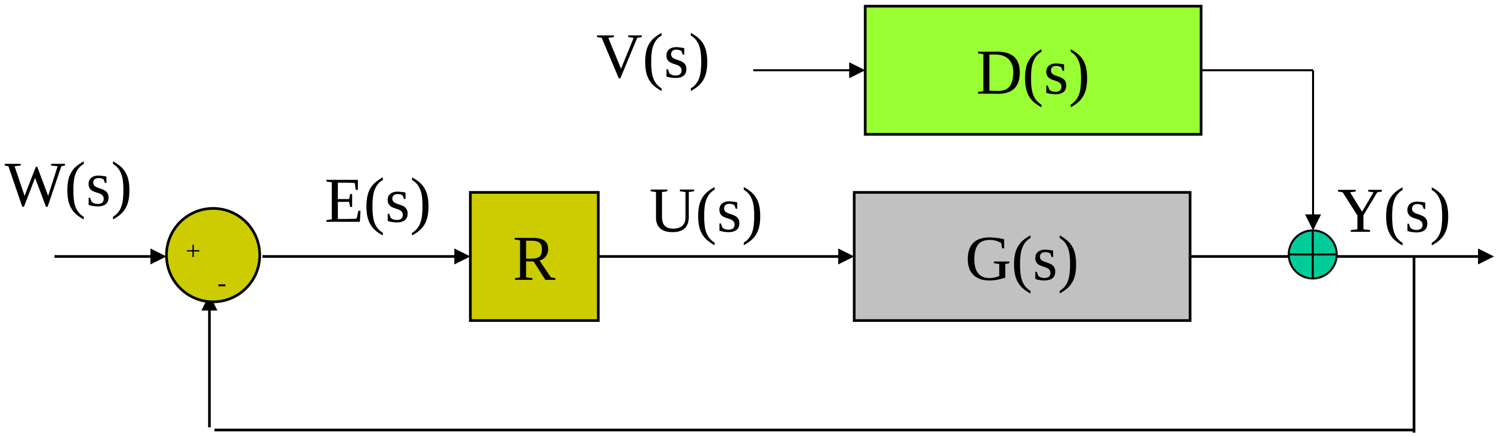
$$e_{ss} = \frac{w}{sG(0)R(0)}$$

If G(s) or R(s) do not have an integrator: Infinite error. If they have one: finite error.

Two integrators are required in G(s)R(s) in order to make the error zero ay steady state.

$$\frac{\overline{GR}(s)}{s} \qquad e_{ss} = \frac{w}{\overline{GR}(0)}$$

4 basic TF

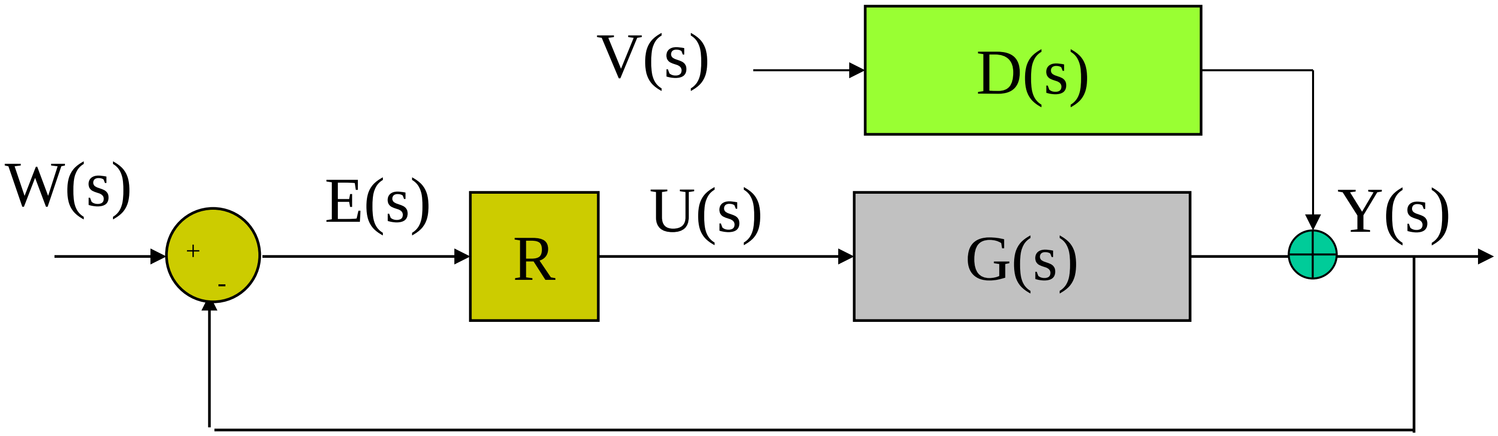


$$Y(s) = \frac{G(s)R(s)}{1+G(s)R(s)} W(s) + \frac{D(s)}{1+G(s)R(s)} V(s)$$

$$U(s) = \frac{R(s)}{1+G(s)R(s)} W(s) + \frac{R(s)D(s)}{1+G(s)R(s)} V(s)$$

It is important to pay attention also to the control efforts

One degree of freedom controllers



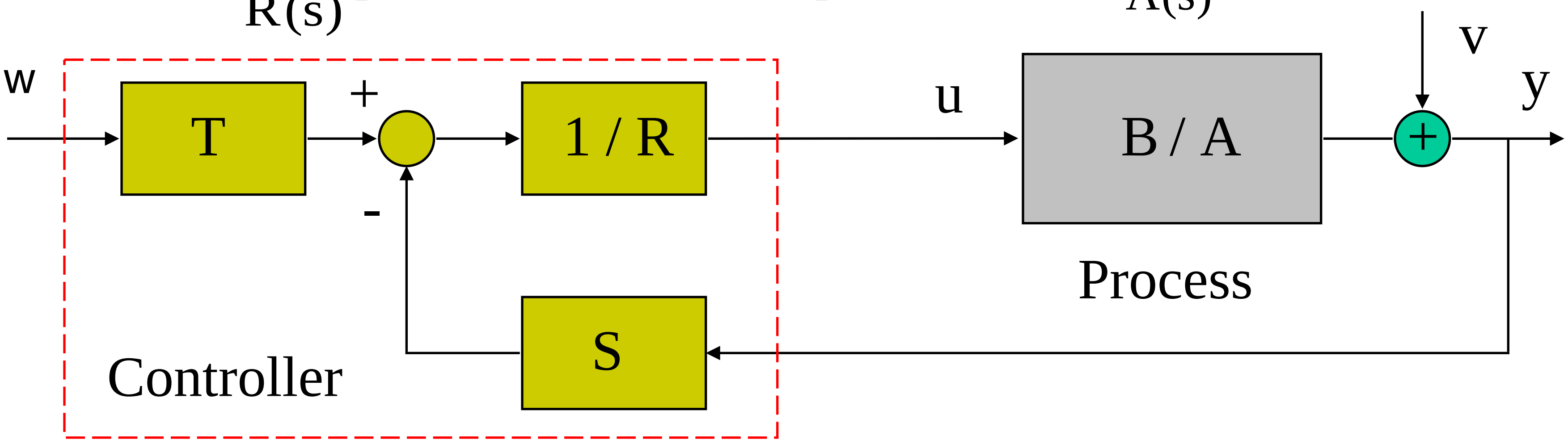
$$Y(s) = \frac{G(s)R(s)}{1+G(s)R(s)}W(s) + \frac{D(s)}{1+G(s)R(s)}V(s)$$

If R(s) is chosen in order to get a good dynamic response against set point changes, then, the response against disturbances is given, and vice-versa. There is no enough degrees of freedom to design the controller for the two aims simultaneously.

Two degrees of freedom controllers 2DOF

$$U(s) = \frac{1}{R(s)}\left[T(s)W(s) - S(s)Y(s)\right]$$

$$Y(s) = \frac{B(s)}{A(s)}U(s) + V(s)$$



$$Y(s) = \frac{B(s)T(s)}{R(s)A(s) + B(s)S(s)}W(s) + \frac{B(s)}{R(s)A(s) + B(s)S(s)}V(s)$$

It is possible to select R and S in order to get a good response against disturbances and select T in order to tune the response against set point changes